



SARDAR PATEL UNIVERSITY

Bachelor of Science (Semester 5) Examination – 2025

US05CMTH54 – Metric Spaces and Topological Spaces

Date: 08/10/2025

Time: 10:00 A.M. To 01:00 P.M.

Total: 70 Marks

Q-1. Multiple Choice Questions

[10]

- (1) On which of the following domain D , the function $d(x, y) = \sin |x - y|$ is metric?
- A. $D = \mathbb{R} \times \mathbb{R}$
B. $D = [0, \infty) \times [0, \infty)$
C. $D = \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$
D. $D = [-2\pi, 2\pi]$
- (2) In $\mathbb{R}$ with usual metric, $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = \underline{\hspace{2cm}}$.
- A. 2
B. 3
C. 1
D. Doesn't exist
- (3) The function $f: (\mathbb{R}, d_1) \rightarrow (\mathbb{R}, d_2)$ defined by $f(x) = \frac{1}{x^2 - 1}$ is continuous on R if,
- A. d_1 is usual metric
B. d_2 is usual metric
C. d_1 is discrete metric
D. d_2 is discrete metric
- (4) In $\mathbb{R}$ with usual metric, the set of all interior points of the set $[-1, 1]$ is
- A. $(-1, 1)$
B. $[-1, 1]$
C. ϕ
D. $\mathbb{R}$
- (5) Which of the following set is open in any metric space?
- A. $\mathbb{N}$
B. $[1, 3] \cup (0, 1)$
C. $[-1, 1] \cap [2, 3]$
D. $\{1\}$
- (6) Which of the following set is open set in $\mathbb{R}$ with usual topology?
- A. $\mathbb{Q}$
B. $\{x \in \mathbb{R} : -2 \leq x \leq 2\}$
C. $[5, 7]$
D. $\{x \in \mathbb{R} : -2 < x < 2\}$
- (7) Let X be any non-empty set. $\tau_1 =$ Discrete topology, $\tau_2 =$ Indiscrete topology. Then,
- A. τ_1 is finer than τ_2
B. τ_2 is finer than τ_1
C. Both are not comparable
D. None of these
- (8) Which of the following is NOT always true?
- A. $A \subset \bar{A}$
B. $\text{Int}(A) \subset A$
C. $\text{Int}(A)$ is open set
D. $\bar{A}$ is open set
- (9) In any topological space (X, τ) , if there is a non-empty proper subset of X which is both τ -open and τ -closed then
- A. X is compact
B. X is connected
C. X is disconnected
D. None
- (10) Let A and B are two nonempty subsets of the topological space (X, τ) . Which of the following condition is there in the definition of disconnected space?
- A. $A \cap B = \phi$
B. $A \cup B = \phi$
C. $A \cap \bar{B} = \phi$
D. $\bar{A} \subset \bar{B}$

[P. T. O.]

(1)

Q-2. Short Answer Questions (Attempt any TEN)**[20]**

- (1) Give any one example of metric that can be defined on $\mathbb{R}^n$.
- (2) For any metric space $(\mathbb{R}^2, d)$, if we define $d(x, y) = |x_1 - x_2| + |y_1 - y_2|$ where $x = (x_1, y_1)$ & $y = (x_2, y_2)$ then prove that d satisfies the triangular inequality.
- (3) Define: Continuous function in metric space.
- (4) In $\mathbb{R}$ with usual metric, is a sequence $x_n = \left\{1 + \frac{1}{n}\right\}$ convergent? If yes, then what is the point of convergence?
- (5) Define: Limit point in metric space.
- (6) Prove that any finite subset of metric space is closed set.
- (7) Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Then write all open sets of (X, τ) .
- (8) Is lower limit topology finer topology than usual topology on $\mathbb{R}$? Justify your answer.
- (9) Define: Interior point in Topological Space.
- (10) Prove that any space with more than one point with discrete topology is disconnected.
- (11) Find set of all cluster points of the set $\{x \in \mathbb{R} : |x| < 1\}$ in $\mathbb{R}$ with usual topology.
- (12) Define: Open covering.

Q-3 (a) Let $d: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ be defined as $d(x, y) = \left[\sum_{k=1}^n (x_k - y_k)^2\right]^{\frac{1}{2}}$ where $x = (x_1, x_2, \dots, x_n)$ & $y = (y_1, y_2, \dots, y_n)$. Then prove that d is a metric on $\mathbb{R}^n$. **[05]**

(b) Prove that a sequence in metric space cannot converges to two different points. **[05]**

OR

Q-3 (a) Let $\rho: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $\rho(x, y) = |x - y|$. Then prove that ρ is metric on $\mathbb{R}$. **[05]**

(b) Prove that every convergent sequence in a metric space is Cauchy. Is converse true? Justify! **[05]**

Q-4 (a) If f and g are continuous function from a metric space (M_1, ρ_1) into metric space (M_2, ρ_2) , then prove that $f * g$ is also continuous on M_1 . **[05]**

(b) Prove that the real valued function f is continuous at $a \in \mathbb{R}$ if and only if the inverse image under f of any open ball $B[f(a), \epsilon]$ about $f(a)$ contains an open ball $B[a, \delta]$ about a . **[05]**

OR

Q-4 (a) Prove that a subset G of the metric space (X, d) is open if and only if compliment of G is closed. **[05]**

(b) Let (X, d) is metric space and let $A \subset B \subset M$. If A is dense in B and if B is dense in M , then prove that A is dense in M . **[05]**

Q-5 (a) Prove that for $a \in \mathbb{R}$, the set $\{a\}$ is closed set in the usual topology, in two different ways. **[05]**

(b) Prove that arbitrary intersection of closed sets is also closed in topological space. **[05]**

OR

Q-5 (a) Prove that discrete topology satisfies all the conditions for becoming a topological space. **[05]**

(b) Let (X, τ) be a topological space and A be a subset of X . A' is the set of all cluster points of A then prove that A is τ -closed if and only if $A' \subset A$. **[05]**

Q-6 (a) Prove that continuous image of connected space is connected. **[05]**

(b) Generate two topologies on a finite set X in which one topology is connected and another topology is disconnected. Justify your answer. **[05]**

OR

Q-6 (a) Let $f: (R, \tau) \rightarrow (R, \psi)$ defined by $f(x) = 2, \forall x \in R$. Then prove that f is $\tau - \psi$ continuous. **[05]**

(b) Assuming that connectedness is a topological property, prove that $\mathbb{R}$ with usual topology and $\mathbb{R}$ with lower limit topology are non-homeomorphic. **[05]**

X
(2)