T.Y.B.Sc. : Semester - V

US05CMTH22(T)

Theory Of Real Functions

[Syllabus effective from June , 2020]

Study Material Prepared by :
Mr. Rajesh P. Solanki
Department of Mathematics and Statistics
V.P. and R.P.T.P. Science College, Vallabh Vidyanagar

US05CMTH22(T)- UNIT : IV

PART-1

1. Extreme Values

Extreme Values

Let f be a function containing (a, b). Then $f(a, b)$ is an extreme value of f if for every (x, y) other than (a, b), of some neighbourhood of (a, b), the difference

$$f(x, y) - f(a, b)$$

keeps the same sign.

2. State the sufficient conditions for a function $z = f(x, y)$ to attain extreme values at a point (a, b)

Sufficient conditions for a function $z = f(x, y)$ to attain extreme.

The sufficient condition for a function $z = f(x, y)$ to attain extreme values at a point (a, b) is that

$$f_x(a, b) = 0 = f_y(a, b) \quad \text{and} \quad f_{xx}(a, b).f_{yy}(a, b) - [f_{xy}(a, b)]^2 > 0$$

3. State the necessary conditions for a function $z = f(x, y)$ to attain extreme values at a point (a, b)

Necessary conditions for a function $z = f(x, y)$ to attain extreme value.

The necessary conditions for a function $z = f(x, y)$ to attain extreme values at a point (a, b) are as follows

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0$$

4. **show that $2x^4 - 3x^2y + y^2$ has neither a maximum nor a minimum at $(0, 0)$.**

Answer:

Let $f(x, y) = 2x^4 - 3x^2y + y^2$.

Then, $f_x(x, y) = 8x^3 - 6xy$ and $f_y(x, y) = -3x^2 + 2y$

and $f_{xx}(x, y) = 24x^2 - 6y$, $f_{xy}(x, y) = -6x$ and $f_{yy}(x, y) = 2$

Now, $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$

$f_{xx}(0, 0) = 0$, $f_{xy}(0, 0) = 0$ and $f_{yy}(0, 0) = 2$

Therefore, $f_{xx}(0, 0).f_{yy}(0, 0) - (f_{xy}(0, 0))^2 = 0$

So, we need further investigation.

Here,

$$\begin{aligned} f(x, y) - f(0, 0) &= (2x^4 - 3x^2y + y^2) - 0 \\ &= (x^2 - y)(2x^2 - y) \end{aligned}$$

Therefore,

$$f(x, y) - f(0, 0) > 0 \ \text{ if } \ y < 0 \ \text{ or } x^2 > y > 0$$

and

$$f(x, y) - f(0, 0) < 0 \ \text{ if } \ y > x^2 > \frac{y}{2} > 0$$

Thus, $f(x, y) - f(0, 0)$ does not keep the same sign near the origin, Hence, f has neither a minimum nor a maximum at the origin.

5. **Show that $y^2 + x^2y + x^4$ has a minimum at $(0, 0)$.**

Answer:

Let $f(x, y) = y^2 + x^2y + x^4$.

We have,

$$\begin{aligned} f(x,y)-f(0,0) &= (y^2+x^2y+x^4)-0 \\ &= y^2+x^2y+\frac{x^4}{4}+\frac{3x^4}{4} \\ &= (y+\frac{x^2}{2})^2+\frac{3x^4}{4} \\ &> 0; \quad \text{when} \quad (x,y)\neq(0,0) \end{aligned}$$

Therefore, $y^2+x^2y+x^4$ has a minimum at $(0,0)$.

6. **Investigate the maxima and minima of the function** $f(x,y)=x^3+y^3-63(x+y)+12xy$

Answer:
For $f(x,y)=x^3+y^3-63(x+y)+12xy$, we have
$f_x(x,y)=3x^2-63+12y,\ f_y(x,y)=3y^2-63+12x.$
For extreme values of f we must have, $f_x(x,y)=0=f_y(x,y)$.
Therefore,
$3x^2-63+12y=0$ - - - - (i) and
$3y^2-63+12x=0.$- - - - (ii)

Subtracting (ii) from (i) we get,

$$\begin{aligned} &3(x^2-y^2)+12(y-x)=0 \\ \Rightarrow&(x-y)(x+y-4)=0 \\ \Rightarrow&x=y \quad \text{or} \quad x=4-y. \end{aligned}$$

Putting $x=y$ in (ii) we get,

$$y^2+4y-21=0 \Rightarrow y=-7 \quad \text{or} \quad y=3.$$

Therefore, $(3,3)$ and $(-7,-7)$ are stationary points of f. Again, putting $x=4-y$ in (ii) ,we get

$3y^2+12(4-y)-63=0$
$3y^2-12y-15=0$
$y^2-4y-5=0$
$(y-5)(y+1)=0$
Therefore, $y=5$ or $y=-1$
We get corresponding stationary points $(5,-1)$ and $(-1,5)$. Thus, $(3,3),(-7,-7),(5,-1)$ and $(-1,5)$ are stationary points of f.
Now,

$$\begin{aligned} A=f_{xx}(x,y)&=6x, \quad C=f_{yy}(x,y)=6y \\ \text{and } B=f_{yx}(x,y)&=f_{xy}(x,y)=12. \end{aligned}$$

At $(3,3)$ we get, $A = 18$, $B = 12$ and $C = 18$,
so that $AC - B^2 = 180 > 0$ and $A > 0$.
Hence $(3,3)$ is a minimum of f and the minimum value of f is
$f(3,3) = 3^3 + 3^3 - 63(3+3) + 12(3)(3) = -216$.

At $(-7,-7)$ we get, $A = -42$, $B = 12$ and $C = -42$,
so that $AC - B^2 = 1620 > 0$ and $A < 0$.
Hence $(-7,-7)$ is a maximum of f and the maximum value of f is
$f(-7,-7) = (-7)^3 + (-7)^3 - 63(-7-7) + 12(-7)(-7) = 784$.

At $(5,-1)$ we get, $A = 30$, $B = 12$, $C = -6$,
so that $AC - B^2 = -324 < 0$.
Hence $(5,-1)$ is not an extreme point of f.
At $(-1,5)$ we get, $A = -6$, $B = 12$, $C = 30$,
so that $AC - B^2 = -324 < 0$.
Hence $(-1,5)$ is not an extreme point of f.

7. **A rectangular box open at the top is to have a volume of $32m^3$. Find the dimensions of box so that the total surface area is minimum.**

Proof:
Suppose lengths of the edges of the box are x, y and z.
Then volume of the box is given by

$$V = xyz$$

,
But, it is given that $V = 32$
Therefore, $32 = xyz$
Therefore, $z = \frac{32}{xy}$

Now, the total surface area of the box, is $2xy + 2yz + 2zx$.
The surface area of the open box is $xy + 2yz + 2zx = xy + 2y\frac{32}{xy} + 2\frac{32}{xy}x = xy + 64(\frac{1}{x} + \frac{1}{y})$.
Let $f(x,y) = xy + 64(\frac{1}{x} + \frac{1}{y})$.

Now,

$$f_x(x,y) = y - \frac{64}{x^2}, \quad f_y(x,y) = x - \frac{64}{y^2}.$$

For extreme values of f we must have, $f_x(x,y) = f_y(x,y) = 0$.
Then $x^2y - 64 = 0 = xy^2 - 64$.
Therefore, $xy(x-y) = 0$.
Hence $x = y$ or $x = 0$ or $y = 0$.
Since any length of each edge must be positive, $x \neq 0, y \neq 0$.
So, we must have, $x = y$.
Taking $x = y$ in $x^2y - 64 = 0$, we get $y^3 = 64$.
That is, $x = y = 4$.
Now $A = f_{xx}(x,y) = \frac{128}{x^3}$, $B = f_{xy}(x,y) = 1$ and $C = f_{yy}(x,y) = \frac{128}{y^3}$.
Therefore at $(4,4)$ we have,
$A = 2$, $B = 1$ and $C = 2$.
Here, $AC - B^2 = 3 > 0$ and $A > 0$.

Therefore, at $(4,4)$ f is minimum.
Thus, for minimum surface area, the edges of the box are

$x = 4m$, $y = 4m$ and $z = \frac{32}{4\times 4} = 2m$.

8. **Show that** $(y-x)^4 + (x-2)^4$ **has a minimum at** $(2,2)$**.**

Answer:
Let $f(x,y) = (y-x)^4 + (x-2)^4$.

Now,

$$\begin{aligned} f(x,y) - f(2,2) &= (y-x)^4 + (x-2)^4 - 0 \\ &= (y-x)^4 + (x-2)^4 \\ &> 0 \ \forall (x,y) \neq (2,2) \\ \therefore f(x,y) - f(2,2) &> 0 \end{aligned}$$

Therefore, $f(x,y)$ has a minimum at $(2,2)$.