S.Y.B.Sc. : Semester - III

US03CMTH21

Numerical Methods

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US03CMTH21- UNIT : IV

1. Derive the general formula for numerical integration

Answer:

Let $y = f(x)$ be a function defined on $[a, b]$. To evaluate the definte integral $\int_a^b y.dx$ let us divide $[a, b]$ into n subintervals $[x_{i-1}, x_i],\ i = 1, 2, \dots, n$ of equal length $h = \dfrac{b-a}{n}$, where,

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

Clearly,

$$x_r = x_0 + rh,\ r = 0, 1, 2, \dots, n$$

Therefore, we have

$$\int_a^b y.dx = \int_{x_0}^{x_n} y.dx \text{ - - - } (1)$$

Now, for $x = x_0 + ph$ corresponding y is given by Newton's forward difference interpolation formula,

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots$$

Integrating both the sides over $[x_0, x_n]$ with respect to x we get,

$$\int_{x_0}^{x_n} y.dx = \int_{x_0}^{x_n} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots\right].dx$$

Since, $x = x_0 + ph$, we have $dx = h.dp$. Also,

$$x = x_0 \Rightarrow p = 0 \quad \text{and} \quad x = x_n \Rightarrow p = n$$

Therefore, integrating (2) by substitution, we get,

$$\int_{x_0}^{x_n} y.dx = h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots\right].dp$$

$$= h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p^2-p}{2}\Delta^2 y_0 + \frac{p^3-3p^2+2p}{6}\Delta^3 y_0 + \dots\right].dp$$

$$= h\left[py_0 + \frac{p^2}{2}\Delta y_0 + \frac{1}{2}\left[\frac{p^3}{3} - \frac{p^2}{2}\right]\Delta^2 y_0 + \frac{1}{6}\left[\frac{p^4}{4} - p^3 + p^2\right]\Delta^3 y_0 + \dots\right]_{p=0}^{p=n}$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left[\frac{n^3}{3} - \frac{n^2}{2}\right]\Delta^2 y_0 + \frac{1}{24}(n^4 - 4n^3 + 4n^2)\Delta^3 y_0 \dots\right]$$

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{1}{2}\left[\frac{n^2}{3} - \frac{n}{2}\right]\Delta^2 y_0 + \frac{n}{24}(n-2)^2\Delta^3 y_0 \dots\right]$$

Therefore,

$$\int_{x_0}^{x_n} y.dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n}{24}(n-2)^2\Delta^3 y_0 \dots\right]$$

This is the general formula for numerical integration of $y = f(x)$ over the interval $[x_0, x_n]$.

2. Derive the general formula for Trapezoidal rule

Answer:

Let $y = f(x)$ be a function defined on $[a, b]$. To evaluate the definte integral $\int_a^b y.dx$ let us divide $[a, b]$ into n subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \dots, n$, all of equal length $h = \frac{b-a}{n}$, where,

$$a = x_0 < x_1 < x_2 < \dots < x_n = b$$

Clearly,

$$x_r = x_0 + rh,\ r = 0, 1, 2, \dots, n$$

Now, the general formula for numerical integration of y over $[x_0, x_n]$ is given by

$$\int_{x_0}^{x_n} y.dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \dots\right]$$

If we set $n = 1$ then we have,

$$\int_{x_0}^{x_1} y.dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$\therefore \int_{x_0}^{x_1} y.dx = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

$$\therefore \int_{x_0}^{x_1} y.dx = \frac{h}{2}(y_0 + y_1)$$

Similarly for the next subinterval, $[x_1, x_2]$, we get,

$$\int_{x_1}^{x_2} y.dx = \frac{h}{2}(y_1 + y_2)$$

In general for any subinterval $[x_{i-1}, x_i]$, we get,

$$\int_{x_{i-1}}^{x_i} y.dx = \frac{h}{2}(y_{i-1} + y_i)$$

Now,

$$\begin{aligned}
\int_{x_0}^{x_n} y.dx &= \int_{x_0}^{x_1} y.dx + \int_{x_1}^{x_2} y.dx + \cdots + \int_{x_{n-1}}^{x_n} y.dx \\
&= \frac{h}{2}\left[(y_0 + y_1) + (y_1 + y_2) + (y_2 + y_3) + \cdots + (y_{n-1} + y_n)\right] \\
\therefore \int_{x_0}^{x_n} y.dx &= \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_2 + \cdots + y_{n-1}) + y_n\right]
\end{aligned}$$

This formula is known as the Trapazoidal Rule for numerical integration.

3. Derive the formula of Simpson's $\frac{1}{3}$-rule for numerical integration.

Answer:

Let $y = f(x)$ be a function defined on $[a, b]$. To evaluate the definte integral $\int_a^b y.dx$ let us divide $[a, b]$ into n subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, all of equal length $h = \frac{b-a}{n}$, where, n is an even positive integer and

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

Clearly,

$$x_r = x_0 + rh,\ r = 0, 1, 2, \ldots, n$$

Now, the general formula for numerical integration of y over $[x_0, x_n]$ is given by

$$\int_{x_0}^{x_n} y.dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \ldots\right]$$

If we set $n = 2$ then we have,

$$\int_{x_0}^{x_2} y.dx = 2h\left[y_0 + \frac{2}{2}\Delta y_0 + \frac{2(4-3)}{12}\Delta^2 y_0\right]$$

$$\therefore \int_{x_0}^{x_2} y.dx = 2h\left[y_0 + (y_1 - y_0) + \frac{1}{6}(y_2 - 2y_1 + y_0)\right]$$

$$\therefore \int_{x_0}^{x_2} y.dx = \frac{2h}{6}[6y_1 + (y_2 - 2y_1 + y_0)]$$

$$\therefore \int_{x_0}^{x_2} y.dx = \frac{h}{3}[y_0 + 4y_1 + y_2]$$

Similarly for the next subinterval, $[x_2, x_4]$, we get,

$$\int_{x_2}^{x_4} y.dx = \frac{h}{3}[y_2 + 4y_3 + y_4]$$

In general for any subinterval $[x_{i-1}, x_{i+1}]$, we get,

$$\int_{x_{i-1}}^{x_{i+1}} y.dx = \frac{h}{2}(y_{i-1} + 4y_i + y_{i+1})$$

Now,

$$\int_{x_0}^{x_n} y.dx = \int_{x_0}^{x_2} y.dx + \int_{x_2}^{x_4} y.dx + \cdots + \int_{x_{n-2}}^{x_n} y.dx$$

$$= \frac{h}{2}\left[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + (y_4 + 4y_5 + y_6) + \cdots + (y_{n-2} + 4y_{n-1} + y_n)\right]$$

$$\therefore \int_{x_0}^{x_n} y.dx = \frac{h}{2}\left[y_0 + 2(y_2 + y_4 + y_6 + \cdots + y_{n-2}) + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + y_n\right]$$

This formula is known as the Simpson's $\frac{1}{3}$-rule for numerical integration.

> **4. Using Newton's forward difference formula, find the general formula for numerical integration and hence derive Simpson's $\frac{3}{8}$-rule**

Answer:

Let $y = f(x)$ be a function defined on $[a, b]$. To evaluate the definte integral $\int\limits_a^b y.dx$ let us divide $[a, b]$ into n subintervals $[x_{i-1}, x_i],\ i = 1, 2, \ldots, n$, all of equal length $h = \dfrac{b-a}{n}$, where, n is a multiple of 3 and

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

Clearly,

$$x_r = x_0 + rh,\ r = 0, 1, 2, \ldots, n$$

Now, the general formula for numerical integration of y over $[x_0, x_n]$ is given by

$$\int\limits_{x_0}^{x_n} y.dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n}{24}(n-2)^2\Delta^3 y_0 \ldots\right]$$

If we set $n = 3$ then we have,

$$\int\limits_{x_0}^{x_3} y.dx = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3(6-3)}{12}\Delta^2 y_0 + \frac{3}{24}(3-2)^2\Delta^3 y_0\right]$$

$$\therefore \int\limits_{x_0}^{x_3} y.dx = 3h\left[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - 2y_1 + y_0) + \frac{1}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$\therefore \int\limits_{x_0}^{x_3} y.dx = \frac{3h}{8}[8y_0 + (12y_1 - 12y_0) + (6y_2 - 12y_1 + 6y_0) + (y_3 - 3y_2 + 3y_1 - y_0)]$$

$$\therefore \int\limits_{x_0}^{x_3} y.dx = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3]$$

Similarly for the next subinterval, $[x_3, x_6]$, we get,

$$\therefore \int\limits_{x_3}^{x_6} y.dx = \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6]$$

In general for any subinterval $[x_{i-1}, x_{i+2}]$, we get,

$$\int\limits_{x_{i-1}}^{x_{i+2}} y.dx = \frac{3h}{8}(y_{i-1} + 3y_i + 3y_{i+1} + y_{i+2})$$

Now,

$$\int_{x_0}^{x_n} y.dx = \int_{x_0}^{x_3} y.dx + \int_{x_3}^{x_6} y.dx + \cdots + \int_{x_{n-3}}^{x_n} y.dx$$

$$= \frac{3h}{8}\left[(y_0 + +3y_1 + 3y_2 + y_3) + (y_3 + +3y_4 + 3y_5 + y_6) + (y_6 + +3y_7 + 3y_8 + y_9) + \cdots + (y_{n-3} + 3y\right.$$

$$\therefore \int_{x_0}^{x_n} y.dx = \frac{3h}{8}\left[y_0 + 2(y_3 + y_6 + y_9 + \cdots + y_{n-3}) + 3(y_1 + y_2 + y_4 + y_5 \cdots + y_{n-1}) + y_n\right]$$

This formula is known as the Simpson's $\frac{3}{8}$-rule for numerical integration.

5. **Find, from the following table , the area bounded by the curve and the x-axis from $x = 7.47$ to $x = 7.52$, using Trapazoidal rule**

x	**7.47**	**7.48**	**7.49**	**7.50**	**7.51**	**7.52**
$f(x)$	**1.93**	**1.95**	**1.98**	**2.01**	**2.03**	**2.06**

Answer:

x	7.47	7.48	7.49	7.50	7.51	7.52
$f(x)$	1.93	1.95	1.98	2.01	2.03	2.06

Using Trapezoidal rule,

$$\int_a^b y.dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n]$$

we get the integral

$$\int_{7.47}^{7.52} \text{y}.dx = \frac{0.01}{2}[1.93 + 2(+1.95 + 1.98 + 2.01 + 2.03) + 2.06]$$

$$= 0.09965$$

6. **Evaluate $\int_0^1 \frac{1}{1+x}dx$, correct upto three decimal places using Simpson's $\frac{1}{3}$-rule and by taking $h = 0.05$**

Here, a = 0, b = 1, and h = 0.05

Therefore, $n = \dfrac{1-0}{0.05} = 20$

Following is the table of values of x and y with number of sub-intervals n = 20 and length of each subinterval h = 0.05

x	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35
$y = \frac{1}{x+1}$	1	0.95238	0.90909	0.86957	0.83333	0.8	0.76923	0.74074
x	0.4	0.45	0.5	0.55	0.6	0.65	0.7	0.75
$y = \frac{1}{x+1}$	0.71429	0.68966	0.66667	0.64516	0.625	0.60606	0.58824	0.57143
x	0.8	0.85	0.9	0.95	1			
$y = \frac{1}{x+1}$	0.55556	0.54054	0.52632	0.51282	0.5			

Using Simpson's $\frac{1}{3}$ rule,

$$\int_a^b y.dx = \frac{h}{3}[y_0 + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1} +)y_n]$$

we get the integral

$$\begin{aligned}\int_0^1 \frac{1}{x+1}.dx = \frac{0.05}{3}[1 &+ 2(0.90909 + 0.83333 + 0.76923 + 0.71429 + 0.66667 + 0.625 + 0.58824 \\ &+ 0.55556 + 0.52632) + \\ &+ 4(0.95238 + 0.86957 + 0.8 + 0.74074 + 0.68966 + 0.64516 + 0.60606 \\ &+ 0.57143 + 0.540540.51282) + 0.5] \\ = 0.69315\end{aligned}$$

7. Find the value of $\int_3^7 x^2 \log x \, dx$ by taking the length of the interval $h = 1$

Here, a = 3, b = 7, and h = 1

Therefore, $n = \dfrac{7-3}{1} = 4$

Following is the table of values of x and y with number of sub-intervals n = 4 and length of each subinterval h = 1

x	3	4	5	6	7
$y = x^2 \log(x)$	9.88751	22.18071	40.23595	64.50334	95.3496

Using Simpson's $\frac{1}{3}$ rule,

$$\int_a^b y.dx = \frac{h}{3}[y_0 + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1} +)y_n]$$

we get the integral

$$\int_3^7 x^2 \log(x)\,.dx = \frac{1}{3}[9.88751 + 2(40.23595) + 4(22.1807164.50334) + 95.3496]$$

$$\int_3^7 x^2 \log(x)\,.dx = 177.48174$$

8. **Evaluate $\int_1^3 \frac{1}{x}\,dx$, by using Simpson's $\frac{1}{3}$ rule with 4 strips.**

Here, a = 1, b = 3, and n = 4
Therefore, $h = \dfrac{3-1}{4} = 0.5$
Following is the table of values of x and y with number of sub-intervals n = 4 and length of each subinterval h = 0.5

x	1	1.5	2	2.5	3
$y = \frac{1}{x}$	1	0.66667	0.5	0.4	0.33333

Using Simpson's $\frac{1}{3}$ rule,

$$\int_a^b y.dx = \frac{h}{3}[y_0 + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1} +)y_n]$$

we get the integral

$$\int_1^3 \frac{1}{x}.dx = \frac{0.5}{3}[1 + 2(0.5) + 4(0.666670.4) + 0.33333]$$

Therefore,

$$\int_1^3 \frac{1}{x}.dx = 1.1$$

9. Find the value of $\int_0^1 \frac{dx}{1+x^2}\ dx$ by taking eight sub intervals using Trapezoidal rule. Also find its error.

Here, a = 0, b = 1, and n = 8
Therefore, $h = \frac{1-0}{8} = 0.125$
Following is the table of values of x and y with number of sub-intervals n = 8 and length of each subinterval h = 0.125

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
$y = \frac{1}{x^2+1}$	1	0.98462	0.94118	0.87671	0.8	0.7191	0.64	0.56637	0.5

Using Simpson's $\frac{1}{3}$ rule,

$$\int_a^b y.dx = \frac{h}{3}[y_0 + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1}+)y_n]$$

we get the integral

$$\int_0^1 \frac{1}{x^2+1}.dx = \frac{0.125}{3}[1 + 2(0.94118 + 0.8 + 0.64) + 4(0.98462 + 0.87671 + 0.71910.56637) + 0.5]$$

Therefore,

$$\int_0^1 \frac{1}{x^2+1}.dx = 0.7854$$

10. Describe the Romberg's Integration method.

Answer:
The Rombreg's method can be used to improve numerical integrals obtained using the finite difference methods. In the following we describe the method to improve numerical integrals obtained for

$$I = \int_a^b y.dx \text{ - - - } (1)$$

by Tapezoidal rule with two different lengths.

Suppose, corrsponding to two different lengths h_1 and h_2 of subintervals of $[a, b]$ the integrals obtained by Trapezoidal rule are I_1 and I_2 with errors E_1 and E_2 respectively.

Therefore,

$$I = I_1 - E_1 = I_2 - E_2$$

Here, the errors are given by

$$E_1 = -\frac{1}{12}(b-a)h_1^2 y''(\overline{x}) \quad \text{and} \quad E_2 = -\frac{1}{12}(b-a)h_2^2 y''(\overline{\overline{x}})$$

Where, $y''(\overline{x})$ and $y''(\overline{\overline{x}})$ are the largest values of $y''(x)$ with respective subintervals. Clearly $y''(\overline{x})$ and $y''(\overline{\overline{x}})$ are nearly equal. Therefore, we get,

$$\frac{E_2}{E_1} \approx \frac{h_2^2}{h_1^2}$$

$$\therefore \frac{E_2}{E_2 - E_1} \approx \frac{h_2^2}{h_2^2 - h_1^2}$$

$$\therefore E_2 \approx \frac{h_2^2}{h_2^2 - h_1^2}(E_2 - E_1)$$

Since, $E_2 - E_1 = I_2 - I_1$ we get,

$$\therefore E_2 \approx \frac{h_2^2}{h_2^2 - h_1^2}(I_2 - I_1)$$

With this approximation in error E_2 we obtain new approximation of integral I by,

$$I_3 = I_2 - E_2 = I_2 - \frac{h_2^2}{h_2^2 - h_1^2}(I_2 - I_1) = \frac{I_2 h_2^2 - I_2 h_1^2 - I_2 h_2^2 + I_1 h_2^2}{h_2^2 - h_1^2}$$

$$I_3 = \frac{I_1 h_2^2 - I_2 h_1^2}{h_2^2 - h_1^2}$$

Provided the errors are of same sign and decrease monotonically, I_3 is closer to actual value I.

Now, if we take $h_2 = \dfrac{h}{2}$ and $h_1 = h$ then we get,

$$\begin{aligned} I_3 &= \frac{I_1\frac{h^2}{4} - I_2h^2}{\frac{h^2}{4} - h^2} \\ &= \frac{\frac{I_1}{4} - I_2}{\frac{1}{4} - 1} \\ &= \frac{\frac{I_1-4I_2}{4}}{\frac{1-4}{4}} \\ \therefore I_3 &= \frac{1}{3}(4I_2 - I_1) \end{aligned}$$

If we use the notations $I(h)$, $I\left(\frac{h}{2}\right)$ and $I\left(h, \frac{h}{2}\right)$ for I_1, I_2 and I_3 respectively then

$$I\left(h, \frac{h}{2}\right) = \frac{1}{3}\left(4I\left(\frac{h}{2}\right) - I(h)\right)$$

We can further improve the results using this form of formula by taking smaller interval lengths and obtain required accuracy of the integral.

11. Using Romberg's method, compute $I = \int_0^1 \dfrac{1}{1+x^2}\, dx$, correct upto four decimal places

Let $y = f(x) = \dfrac{1}{1+x^2}$

Initially we take $n = 2$
$\therefore h = \frac{1-0}{2} = 0.5$ and table corresponding table is

x	0	0.5	1
y	1	0.8	0.5

$\therefore$ using Trapezoidal rule

$$I(h) = \frac{h}{2}[y_0 + 2y_1 + y_2]$$

$$\therefore I(h) = \frac{0.5}{2}[1 + 2(0.8) + 0.5]$$

$$\therefore I(h) = 0.775$$

Next, we shall take $h_1 = \frac{h}{2} = \frac{0.5}{2} = 0.25$ Therefore, corresponding table is

x	0	0.25	0.5	0.75	1
y	1	0.94117647	0.64	0.8	0.5

$\therefore$ using Trapezoidal rule

$$I(\tfrac{h}{2}) = \frac{h_1}{2}[y_0 + 2(y_1 + y_2 + y_3) + y_4]$$

$$\therefore I(\tfrac{h}{2}) = \frac{0.25}{2}[1 + 2(0.94117647 + 0.64 + 0.8) + 0.5]$$

$\therefore I(\frac{h}{2}) = 0.782794118$

Next, we shall take $h_2 = \frac{h_1}{2} = \frac{0.25}{2} = 0.125$ Therefore, corresponding table is

x	0	0.125	0.25	0.375	0.5	0.625	0.75	875	1
y	1	0.98461538	0.94117647	0.87671233	0.8	0.71910112	0.64	0.56637168	0.5

$\therefore$ using Trapezoidal rule

$I(\frac{h}{4}) = \frac{h_2}{2}[y_0 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7) + y_4]$

$\therefore I(\frac{h}{4}) = \frac{0.125}{2}[1 + 2(0.98461538 + 0.94117647 + 0.87671233 + 0.8 + 0.71910112 + 0.64 + 0.56637168) + 0.5]$

$\therefore I(\frac{h}{4}) = 0.7846542$

Now, using Romberg's formula

$I(h, \frac{h}{2}) = \frac{1}{3}\left[4I(\frac{h}{2}) - I(h)\right] = \frac{1}{3}[4(0.782794118) - 0.775] = 0.7853922$

Also, using Romberg's formula

$I(\frac{h}{2}, \frac{h}{4}) = \frac{1}{3}\left[4I(\frac{h}{4}) - \frac{h}{2}\right] = \frac{1}{3}[4(0.7846542) - 0.782794118] = 0.78530$

Thus,

$$\int_0^1 \frac{1}{1+x^2}\, dx \approx 0.7853$$

Now,

$$\int_0^1 \frac{1}{1+x^2}\, dx = [tan^{-1}x]_0^1 = tan^{-1}1 - tan^{-1}0 = \frac{\pi}{4} \approx 0.7853981$$

Therefore,

$$Error \approx 0.7853981 - 0.7853 = 0.0000981$$

12. Using Romberg's method, compute $I = \int_0^1 \frac{1}{1+x}\, dx$, correct upto three decimal places

Answer:

Let $y = f(x) = \frac{1}{1+x}$

Initially we take $n = 2$

$\therefore h = \frac{1-0}{2} = 0.5$ and table corresponding table is

x	0	0.5	1
$y = \frac{1}{x+1}$	1	0.6666667	0.5

Using Trapezoidal rule,

$$\int_a^b y.dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n]$$

we get the integral

$$\int_0^1 \frac{1}{x+1}.dx = \frac{0.5}{2}[1.0 + 2(+0.6666667) + 0.5]$$

Therefore,

$$I(h) = 0.7083333$$

Next, we shall take $h_1 = \frac{h}{2} = \frac{0.5}{2} = 0.25$
Corresponding table is

x	0	0.25	0.5	0.75	1
$y = \frac{1}{x+1}$	1	0.8	0.6666667	0.5714286	0.5

Using Trapezoidal rule we get,

$$\int_0^1 \frac{1}{x+1}.dx = \frac{0.25}{2}[1.0 + 2(+0.8 + 0.6666667 + 0.5714286) + 0.5]$$

Therefore,

$$I\left(\frac{h}{2}\right) = 0.6970238$$

Next, we shall take $h_2 = \frac{h_1}{2} = \frac{0.25}{2} = 0.125.$
Corresponding table is

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
$y = \frac{1}{x+1}$	1	0.8888889	0.8	0.7272727	0.6666667	0.6153846	0.5714286	0.5333333	0.5

Using Trapezoidal rule we get the integral

$$\int_0^1 \frac{1}{x+1}.dx = \frac{0.125}{2}[1.0 + 2(+0.8888889 + 0.8 + 0.7272727 + 0.6666667 + 0.6153846$$

$$+ 0.5714286 + 0.5333333) + 0.5]$$

Therefore,

$$I\left(\frac{h}{4}\right) = 0.6941219$$

Now, using Romberg's formula

$I(h, \frac{h}{2}) = \frac{1}{3}\left[4I(\frac{h}{2}) - I(h)\right] = \frac{1}{3}[4(0.6970238) - 0.7083333] = 0.693254$

Also, using Romberg's formula

$I(\frac{h}{2}, \frac{h}{4}) = \frac{1}{3}\left[4I(\frac{h}{4}) - \frac{h}{2}\right] = \frac{1}{3}[4(0.6941219) - 0.6970238] = 0.6931546$

Thus, the integral correct upto 3 decimal places is given by,

$$\int_0^1 \frac{1}{1+x^2}\, dx \approx 0.693$$

> 13. **From the Taylor's series for $y(x)$, find $y(0.1)$ correct upto four decimal places if $y(x)$ satisfies $\frac{dy}{dx} = x - y^2$ and $y(0) = 1$**

Answer:

The Taylor's series for $y(x)$ about a point x_0 is given by

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \frac{(x-x_0)^4}{4!}y^{(iv)}(x_0) + \frac{(x-x_0)^5}{5!}y^{(v)}(x_0) + \ldots$$

For $x_0 = 0$ we get,

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2}y''(0) + \frac{x^3}{6}y'''(0) + \frac{x^4}{24}y^{(iv)}(0) + \frac{x^5}{120}y^{(v)}(0) + \ldots \text{ - - - } (1)$$

We have, $y' = x - y^2$, $x_0 = 0$ and $y(x_0) = y(0) = 1$. Therefore,

$$y'(0) = 0 - y(0)^2 = -1$$

Differentiating $y' = x - y^2$ w.r.t. x we get $y'' = 1 - 2yy'$.

Therefore,

$$y''(0) = 1 - 2y(0)y'(0) = y''(0) = 1 - 2y(0)y'(0) = 1 - 2(1)(-1) = 3$$

Differentiating $y'' = 1 - 2yy'$ w.r.t. x we get $y''' = -2(yy'' + y'^2)$.

Therefore,

$$y'''(0) == -2[y(0)y''(0) + y'^2(0)) = -2[(1)(3) + (-1)^2)] = -8$$

Differentiating $y''' = -2(yy'' + y'^2)$ w.r.t. x we get
$y^{(iv)} = -2(yy''' + y'y'' + 2y'y'') = -2(yy''' + 3y'y'')$.

Therefore,

$$y^{(iv)}(0) = -2(y(0)y'''(0) + 3y'(0)y''(0)) = -2[(1)(-8) + 3(-1)(3)] = 34$$

Differentiating $y^{(iv)} = -2(yy''' + 3y'y'')$ w.r.t. x we get $y^{(v)} = -2(yy^{(iv)} + 4y'y''' + 3y''^2)$

Therefore,
$y^{(v)}(0) = -2(y(0)y^{(iv)}(0) + 3y'(0)y'''(0) + 3y''^2(0)) = -2[(1)(34) + 4(-1)(-8) + 3(3)^2] = -186$
Substituting in (1) we get,

$$y(x) = 1 - x + \frac{3x^2}{2} - \frac{4x^3}{3} + \frac{17x^4}{12} - \frac{31x^5}{20} + \dots$$

Now, we want accuracy of 4 decimal places in the approximation of $y(0.1)$. Therefore any term with its values less than 0.00005 can be decarded and with remaining terms the $y(0.1)$ can be approximated upto 4 decimal places.

Now,

$$\frac{31}{20}x^5 \leqslant 0.00005 \quad \text{whenever} \quad x^5 \leqslant 0.00005 \times \frac{20}{31}$$

$$\therefore \frac{31}{20}x^5 \leqslant 0.00005 \quad \text{whenever} \quad x \leqslant 0.126$$

As $0.1 < 0.126$ it implies that, all the terms from $\frac{31}{20}x^5$ onwards do not contribute upto 4 decimal places to the value of $y(0.1)$. Therefore the value of $y(0.1)$ correct upto 4 deicmal places is given by,

$$y(0.1) \approx 1 - 0.1 + \frac{3(0.1)^2}{2} - \frac{4(0.1)^3}{3} + \frac{17(0.1)^4}{12}$$

$$y(0.1) \approx 0.9138$$

14. Describe Picard's method of successive approximation

Answer:
Let $\frac{dy}{dx} = f(x, y)$ be a differential equation with the initial condition $y(x_0) = y_0$. Integrating the equation, we get the Integral equation,

$$y = y_0 + \int_{x_0}^{x} f(x, y).dx \text{ - - - } (1)$$

To solve the equation using Picard's method we appoximate the solution y successively as follows.

First we replace y in $f(x, y)$ on the RHS of (1) with y_0 so that $f(x, y_0)$ is a function of x

only. Then we integrate it to get approximation $y^{(1)}$ of y as,

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0).dx$$

which is a funcion of x only.

Now, again replace y on the RHS of (1) and then integrate it to obtain next approximation $y^{(2)}$ as,

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)}).dx$$

Continuing similarly we obtain the approcimations $y^{(1)}, y^{(2)}, \ldots y^{(r)}$ using following formula

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)}).dx$$

The sequence converges to y provided certain conditions are satisfied.

15. Use Picard's method to approximate y when $x = 0.25$, given that $y(0) = 0$ and $\frac{dy}{dx} = \frac{x^2}{y^2+1}$ correct upto three decimal places

Here the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2+1}$ is subject to $y(0) = 0$.

Let $x_0 = 0$ and $y_0 = 0$ and $f(x, y) = \frac{x^2}{y^2+1}$. Now we use Picard's method to find the solution as follows.

Therefore the integral equation

$$y = y_0 + \int_{x_0}^{x} f(x, y).dx$$

is given by,

$$y = \int_{0}^{x} \frac{x^2}{y^2+1}.dx \text{ - - - } (1)$$

First we take $y = y_0 = 0$ on the RHS and then integrate it to ontain first approximation.

$$y^{(1)} = \int_{0}^{x} \frac{x^2}{0^2+1}.dx = \frac{x^3}{3}$$

Next replaceing y on the RHS of (1) with $y^{(1)} = \dfrac{x^3}{3}$

$$y^{(2)} = \int_0^x \frac{x^2}{\left(\frac{x^3}{3}\right)^2 + 1}.dx = \tan^{-1}\frac{x^3}{3}$$

Now, we have infinite series expansion of $\tan^{-1} t$ given by

$$tan^{-1}t = t - \frac{t^3}{3} + \frac{t^5}{5} - \cdots + \frac{(-1)^n t^{2n+1}}{2n+1} + \ldots$$

Therefore,

$$y^{(2)} = \frac{x^3}{3} - \frac{\left(\frac{x^3}{3}\right)^3}{3} + \ldots$$

$$\therefore\ y^{(2)} = \frac{x^3}{3} - \frac{x^9}{81} + \ldots \text{ - - - } (2)$$

Now, to determine which terms in the exapansion do not contribute upto three decimal places while evaluating y using (2) we first test the term $\dfrac{x^9}{81}$.

For $\dfrac{x^9}{81} < 0.0005$ we have $x^9 < 0.0005 \times 81$. Therefore for all $x < 0.70028$ the terms of (2) from $\dfrac{x^9}{81}$ do not contribute upto 3 digits after the decimal point. Since, we have to evaluate y at $x = 0.25$ which is less than 0.7 we evaluate $y(0.2)$ with treee digits accuracy as follows,

$$y(0.25) = \frac{0.2^3}{3} = 0.005208333 \approx 0.005$$

16. Use Picard's method to approximate y when $x = 0.2$, given that $y(0) = 1$ and $\dfrac{dy}{dx} = x - y$, correct upto four decimal places

Here, $f(x, y) = x - y,\ x_0 = 0$ and $y_0 = 1$.

Therefore the integral equation

$$y = y_0 + \int_{x_0}^x f(x, y).dx$$

is given by,

$$y = 1 + \int_0^x (x - y).dx \text{ - - - } (1)$$

First we take $y = y_0 = 1$ on the RHS and then integrate it to ontain first approximation.

$$y^{(1)} = 1 + \int_0^x (x - 1).dx = 1 - x + \frac{x^2}{2}$$

Next replaceing y on the RHS of (1) with $y^{(1)} = 1 - x + \frac{x^2}{2}$ we get,

$$y^{(2)} = 1 + \int_0^x \left(x - 1 + x - \frac{x^2}{2}\right).dx = 1 + -x + x^2 - \frac{x^3}{6}$$

Next replaceing y on the RHS of (1) with $y^{(2)} = 1 + -x + x^2 - \frac{x^3}{6}$ we get,

$$y^{(3)} = 1 + \int_0^x \left(x - 1 + x - x^2 + \frac{x^3}{6}\right).dx$$

Therefore,

$$y^{(3)} = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{24}$$

Next replaceing y on the RHS of (1) with $y^{(3)} = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{24}$ we get,

$$y^{(4)} = 1 + \int_0^x x - 1 + x - x^2 + \frac{x^3}{3} - \frac{x^4}{24}$$

Therefore,

$$y^{(4)} = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120}$$

Now,

$$y^{(1)}(0.2) = 1 - (0.2) + \frac{(0.2)^2}{2} = 0.84$$

similarly,

$$y^{(2)}(0.2) = 0.8373333$$
$$y^{(3)}(0.2) = 0.8374666$$
$$y^{(4)}(0.2) = 0.8374640000$$

Therefore, approximation of $y(0.2)$ correct upto 4 decimal places is

$$y(0.2) \approx 0.8374$$

17. Discuss Euler's method for solving a differential equation.

Euler's Method:

Let $\frac{dy}{dx} = f(x, y)$ be a differential equation with the initial condition $y(x_0) = y_0$. Integrating the equation, we get the Integral equation,

$$y = y_0 + \int_{x_0}^{x} f(x, y).dx$$

To solve the equation for $y(x)$ corresponding to some given value of $x = x_0 + rh,\ r = 1, 2, \ldots$ where h is uniform length for each subintervals $[x_{i-1}, x_i]$.

Now, for each $x_0 < x < x_1$, we assume $f(x,y) = f(x_0,y_0)$ and evaluate the integral as follows,

$$\begin{aligned}
y(x_1) &= y_1 \\
&= y_0 + \int_{x_0}^{x_1} f(x_0,y_0).dx \\
&= y_0 + f(x_0,y_0)\int_{x_0}^{x_1} dx \\
&= y_0 + f(x_0,y_0)(x_1 - x_0) \\
\therefore, y(x_1) &= y_0 + hf(x_0,y_0)
\end{aligned}$$

similarly, for each $x_1 < x < x_2$ assuming $f(x,y) = f(x_1,y_1)$ we get,

$$y(x_2) = y_2 = y_1 + hf(x_1,y_1)$$

Proceeding similarly, we obtain the general formula for Euler's method,

$$y(x_{n+1}) = y_{n+1} = y_n + hf(x_n,y_n)$$

The process is very slow to obtain desired accuracy.

18. Dscuss the Euler's Modified method for solving a differential equation.

Euler's Modified Method:

Let $\dfrac{dy}{dx} = f(x,y)$ be a differential equation with the initial condition $y(x_0) = y_0$. Integrating the equation, we get the Integral equation,

$$y = y_0 + \int_{x_0}^{x} f(x,y).dx$$

To solve the equation for $y(x)$ corresponding to some given value of $x = x_0 + rh,\ r = 1,2,\ldots$ where h is uniform length for each subintervals $[x_{i-1}, x_i]$, first we use Euler's original formula to get initial approximation to y_1, say $y_1^{(0)}$ as given below,

$$y_1^{(0)} = y_0 + hf(x_0,y_0)$$

Then, we proceed to improve values of y_1 using trapezoidal rule and take $f(x,y) = \frac{1}{2}[f(x_0,y_0) + f(x_1,y_1^{(0)})]$ and evaluate next approximation of y_1 as follows,

$$y_1^{(1)} = y_0 + \int_{x_0}^{x_1} \frac{1}{2}[f(x_0,y_0) + f(x_1,y_1^{(0)})].dx$$

$$= y_0 + \frac{1}{2}[f(x_0,y_0) + f(x_1,y_1^{(0)})]\int_{x_0}^{x_1} dx$$

$$= y_0 + \frac{1}{2}[f(x_0,y_0) + f(x_1,y_1^{(0)})](x_1 - x_0)$$

$$\therefore, y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1,y_1^{(0)})]$$

Again replacing $y_1^{(0)}$ by $y_1^{(1)}$ on RHS we get next approximation,

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1,y_1^{(1)})]$$

We continue the process similarly to get desired accuracy of y_1.

In general, to approximate y_{r+1} using the same process, first we obtain the initial approximation $y_{r+1}^{(0)}$ using Euler's ordinary formula,

$$y_{r+1}^{(0)} = y_r + hf(x_r, y_r)$$

Then use

$$y_{r+1}^{(n+1)} = y_r + \frac{h}{2}[f(x_r,y_r) + f(x_{r+1},y_{r+1}^{(n)})]$$

repeatedly for $n = 1, 2, \ldots$ to get desired accuracy in y_{r+1}.

19. Using Euler's method solve $y' = -y$, taking five subintervals and $h = 0.01$ with initial condition $y(0) = 1$.

Given differential equation is

$$\frac{dy}{dx} = -y \quad \text{where} \quad y(0) = 1$$

Now, to find $y(0.05)$ let us take $x = 0.05$.Taking h = 0.01, we get

$x_0 = 0,\ x_1 = 0.01,\ x_2 = 0.02,\ x_3 = 0.03,\ x_4 = 0.04,\ x_5 = 0.05,$
Now, we shall find $y_1,\ y_2,\ y_3,\ y_4,\ y_5,\ ,$ using Euler's Modified method.

Finding $\mathbf{y(x_1) = y(0.01)}$

$$\begin{aligned} y_1 = y(0.01) &= y_0 + hf(x_0, y_0) \\ &= 1 + 0.01f(0,1) \\ &= 1 + 0.01[-1] \\ &= 1 + 0.01[-1] \\ y(0.01) &= 0.99 \end{aligned}$$

Finding $\mathbf{y(x_2) = y(0.02)}$

$$\begin{aligned} y_2 = y(0.02) &= y_1 + hf(x_1, y_1) \\ &= 0.99 + 0.01f(0.01, 0.99) \\ &= 0.99 + 0.01[-0.99] \\ &= 0.99 + 0.01[-0.99] \\ y(0.02) &= 0.9801 \end{aligned}$$

Finding $\mathbf{y(x_3) = y(0.03)}$

$$\begin{aligned} y_3 = y(0.03) &= y_2 + hf(x_2, y_2) \\ &= 0.9801 + 0.01f(0.02, 0.9801) \\ &= 0.9801 + 0.01[-0.9801] \\ &= 0.9801 + 0.01[-0.9801] \\ y(0.03) &= 0.970299 \end{aligned}$$

Finding $\mathbf{y(x_4) = y(0.04)}$

$$\begin{aligned} y_4 = y(0.04) &= y_3 + hf(x_3, y_3) \\ &= 0.970299 + 0.01f(0.03, 0.970299) \\ &= 0.970299 + 0.01[-0.970299] \\ &= 0.970299 + 0.01[-0.970299] \\ y(0.04) &= 0.96059601 \end{aligned}$$

Finding $\mathbf{y(x_5) = y(0.05)}$

$$
\begin{aligned}
y_5 = y(0.05) &= y_4 + hf(x_4, y_4)\\
&= 0.96059601 + 0.01f(0.04, 0.96059601)\\
&= 0.96059601 + 0.01[-0.96059601]\\
&= 0.96059601 + 0.01[-0.96059601]\\
y(0.05) &= 0.9509900499
\end{aligned}
$$

20. **Solve $y' = 1 + y^2,\ y(0) = 0$ by Euler's method at the points 0.2 and 0.4**

Given differential equation is

$$\frac{dy}{dx} = 1 + y^2 \quad \text{where} \quad y(0) = 0$$

Now, to find $y(0.2)$ let us take $x = 0.2$.Taking h = 0.05, we get

$x_0 = 0.,\ x_1 = 0.05,\ x_2 = 0.1,\ x_3 = 0.15,\ x_4 = 0.2,$
Now, we shall find $y_1,\ y_2,\ y_3,\ y_4,\ ,$ using Euler's Modified method.

Finding $\mathbf{y(x_1) = y(0.05)}$

$$
\begin{aligned}
y_1 = y(0.05) &= y_0 + hf(x_0, y_0)\\
&= 0 + 0.05f(0, 0)\\
&= 0 + 0.05[1 + 0^2]\\
&= 0 + 0.05[1]\\
y(0.05) &= 0.05
\end{aligned}
$$

Finding $\mathbf{y(x_2) = y(0.1)}$

$$
\begin{aligned}
y_2 = y(0.1) &= y_1 + hf(x_1, y_1)\\
&= 0.05 + 0.05f(0.05, 0.05)\\
&= 0.05 + 0.05[1 + 0.05^2]\\
&= 0.05 + 0.05[1.0025]\\
y(0.1) &= 0.100125
\end{aligned}
$$

Finding $\mathbf{y(x_3) = y(0.15)}$

$$
\begin{aligned}
y_3 = y(0.15) &= y_2 + hf(x_2, y_2)\\
&= 0.100125 + 0.05f(0.1, 0.100125)\\
&= 0.100125 + 0.05[1 + 0.100125^2]\\
&= 0.100125 + 0.05[1.01002502]\\
y(0.15) &= 0.15062625
\end{aligned}
$$

Finding $\mathbf{y(x_4) = y(0.2)}$

$$
\begin{aligned}
y_4 = y(0.2) &= y_3 + hf(x_3, y_3)\\
&= 0.15062625 + 0.05f(0.15, 0.15062625)\\
&= 0.15062625 + 0.05[1 + 0.15062625^2]\\
&= 0.15062625 + 0.05[1.02268827]\\
y(0.2) &= 0.20176066
\end{aligned}
$$

Now, to find $y(0.4)$ let us take $x = 0.4$.Taking h $= 0.05$, we get

$x_0 = 0,\ x_1 = 0.05,\ x_2 = 0.1,\ x_3 = 0.15,\ x_4 = 0.2,\ x_5 = 0.25,\ x_6 = 0.3,\ x_7 = 0.35,\ x_8 = 0.4,$
Now, we shall find $y_1,\ y_2,\ y_3,\ y_4,\ y_5,\ y_6,\ y_7,\ y_8,\ ,$ using Euler's Modified method.

Finding $\mathbf{y(x_1) = y(0.05)}$

$$
\begin{aligned}
y_1 = y(0.05) &= y_0 + hf(x_0, y_0)\\
&= 0 + 0.05f(0, 0)\\
&= 0 + 0.05[1 + 0^2]\\
&= 0 + 0.05[1]\\
y(0.05) &= 0.05
\end{aligned}
$$

Finding $\mathbf{y(x_2) = y(0.1)}$

$$
\begin{aligned}
y_2 = y(0.1) &= y_1 + hf(x_1, y_1)\\
&= 0.05 + 0.05f(0.05, 0.05)\\
&= 0.05 + 0.05[1 + 0.05^2]\\
&= 0.05 + 0.05[1.0025]\\
y(0.1) &= 0.100125
\end{aligned}
$$

Finding $\mathbf{y(x_3) = y(0.15)}$

$$\begin{aligned}
y_3 = y(0.15) &= y_2 + hf(x_2, y_2) \\
&= 0.100125 + 0.05f(0.1, 0.100125) \\
&= 0.100125 + 0.05[1 + 0.100125^2] \\
&= 0.100125 + 0.05[1.01002502] \\
y(0.15) &= 0.15062625
\end{aligned}$$

Finding $\mathbf{y(x_4) = y(0.2)}$

$$\begin{aligned}
y_4 = y(0.2) &= y_3 + hf(x_3, y_3) \\
&= 0.15062625 + 0.05f(0.15, 0.15062625) \\
&= 0.15062625 + 0.05[1 + 0.15062625^2] \\
&= 0.15062625 + 0.05[1.02268827] \\
y(0.2) &= 0.20176066
\end{aligned}$$

Finding $\mathbf{y(x_5) = y(0.25)}$

$$\begin{aligned}
y_5 = y(0.25) &= y_4 + hf(x_4, y_4) \\
&= 0.20176066 + 0.05f(0.2, 0.20176066) \\
&= 0.20176066 + 0.05[1 + 0.20176066^2] \\
&= 0.20176066 + 0.05[1.04070737] \\
y(0.25) &= 0.25379603
\end{aligned}$$

Finding $\mathbf{y(x_6) = y(0.3)}$

$$\begin{aligned}
y_6 = y(0.3) &= y_5 + hf(x_5, y_5) \\
&= 0.25379603 + 0.05f(0.25, 0.25379603) \\
&= 0.25379603 + 0.05[1 + 0.25379603^2] \\
&= 0.25379603 + 0.05[1.06441243] \\
y(0.3) &= 0.30701665
\end{aligned}$$

Finding $\mathbf{y(x_7) = y(0.35)}$

$$
\begin{aligned}
y_7 = y(0.35) &= y_6 + hf(x_6, y_6) \\
&= 0.30701665 + 0.05f(0.3, 0.30701665) \\
&= 0.30701665 + 0.05[1 + 0.30701665^2] \\
&= 0.30701665 + 0.05[1.09425923] \\
y(0.35) &= 0.36172962
\end{aligned}
$$

Finding $\mathbf{y(x_8) = y(0.4)}$

$$
\begin{aligned}
y_8 = y(0.4) &= y_7 + hf(x_7, y_7) \\
&= 0.36172962 + 0.05f(0.35, 0.36172962) \\
&= 0.36172962 + 0.05[1 + 0.36172962^2] \\
&= 0.36172962 + 0.05[1.13084831] \\
y(0.4) &= 0.41827203
\end{aligned}
$$

21. **Given that $\frac{dy}{dx} = x^3 + y,\ y(0) = 1$, determine $y(0.02)$ using Euler's method, , taking $h = 0.01$**

Given differential equation is

$$\frac{dy}{dx} = x^3 + y \quad \text{where} \quad y(0) = 1$$

Now, to find $y(0.02)$ let us take $x = 0.02$.Taking h = 0.01, we get

$x_0 = 0.,\ x_1 = 0.01,\ x_2 = 0.02,$
Now, we shall find $y_1,\ \ y_2,\ \ ,$ using Euler's Modified method.

Finding $\mathbf{y(x_1) = y(0.01)}$

$$
\begin{aligned}
y_1 = y(0.01) &= y_0 + hf(x_0, y_0) \\
&= 1 + 0.01f(0, 1) \\
&= 1 + 0.01[0^3 + 1] \\
&= 1 + 0.01[1] \\
y(0.01) &= 1.01
\end{aligned}
$$

Finding $\mathbf{y(x_2) = y(0.02)}$

$$\begin{aligned} y_2 = y(0.02) &= y_1 + hf(x_1, y_1) \\ &= 1.01 + 0.01 f(0.01, 1.01) \\ &= 1.01 + 0.01[0.01^3 + 1.01] \\ &= 1.01 + 0.01[1.010001] \\ y(0.02) &= 1.02010001 \end{aligned}$$

22. **Given that $\frac{dy}{dx} = x^2 + y,\ y(0) = 1$, determine $y(0.1)$ using Euler's modified method, correct upto four decimal places**

Given differential equation is

$$\frac{dy}{dx} = x^2 + y \quad \text{where} \quad y(0) = 1$$

Now, to find $y(0.1)$ let us take $x = 0.1$.Taking h = 0.05, we get

$x_0 = 0.,\ x_1 = 0.05,\ x_2 = 0.1,$
Now, we shall find $y_0,\ y_1,\ y_2,\ ,$ using Euler's Modified method.

Finding $\mathbf{y(x_1) = y(0.05)}$

We shall find $y_1^{(0)}$ using Euler's Method.

$$\begin{aligned} y_1^{(0)} &= y_0 + hf(x_0, y_0) \\ &= 1 + 0.05 f(0, 1) \\ &= 1 + 0.05[0^2 + 1] \\ &= 1 + 0.05[1] \\ y_1^{(0)} &= 1.05 \end{aligned}$$

Next, we shall use Euler's Modified Formula with $y_1^{(0)}$ to find $y_1^{(1)}$

$$\begin{aligned} y_1^{(1)} &= y_0 + \frac{h}{2}[f(x_0, y_0)]f(x_1, y_1^{(0)} \\ &= 1 + \frac{0.05}{2}[f(0,1) + f(0.05, 1.05)] \\ &= 1 + \frac{0.05}{2} + [0^2 + 1 + 0.05^2 + 1.05] \\ &= 1 + \frac{0.05}{2}[1 + 1.0525] \\ y_1^{(1)} &= 1.0513125 \end{aligned}$$

Next, we shall use Euler's Modified Formula with $y_1^{(0)}$ to find $y_1^{(1)}$

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2}[f(x_0, y_0)]f(x_1, y_1^{(1)} \\ &= 1 + \frac{0.05}{2}[f(0,1) + f(0.05, 1.0513125)] \\ &= 1 + \frac{0.05}{2} + [0^2 + 1 + 0.05^2 + 1.0513125] \\ &= 1 + \frac{0.05}{2}[1 + 1.0538125] \\ y_1^{(2)} &= 1.0513453125 \end{aligned}$$

Therefore,

$$\boxed{\mathbf{y_1 = 1.0513}}$$

Finding $\mathbf{y(x_2) = y(0.1)}$
We shall find $y_2^{(0)}$ using Euler's Method.

$$\begin{aligned} y_2^{(0)} &= y_1 h + f(x_1, y_1) \\ &= 1.0513 + 0.05 f(0.05, 1.0513) \\ &= 1.0513 + 0.05[0.05^2 + 1.0513] \\ &= 1.0513 + 0.05[1.0538] \\ y_2^{(0)} &= 1.10399 \end{aligned}$$

Next, we shall use Euler's Modified Formula with $y_2^{(0)}$ to find $y_2^{(1)}$

$$\begin{aligned}
y_2^{(1)} &= y_1 + \frac{h}{2}[f(x_1, y_1)]f(x_2, y_2^{(0)} \\
&= 1.0513 + \frac{0.05}{2}[f(0.05, 1.0513) + f(0.1, 1.10399)] \\
&= 1.0513 + \frac{0.05}{2} + [0.05^2 + 1.0513 + 0.1^2 + 1.10399] \\
&= 1.0513 + \frac{0.05}{2}[1.0538 + 1.11399] \\
y_2^{(1)} &= 1.10549475
\end{aligned}$$

Next, we shall use Euler's Modified Formula with $y_2^{(0)}$ to find $y_2^{(1)}$

$$\begin{aligned}
y_2^{(2)} &= y_1 + \frac{h}{2}[f(x_1, y_1)]f(x_2, y_2^{(1)} \\
&= 1.0513 + \frac{0.05}{2}[f(0.05, 1.0513) + f(0.1, 1.10549475)] \\
&= 1.0513 + \frac{0.05}{2} + [0.05^2 + 1.0513 + 0.1^2 + 1.10549475] \\
&= 1.0513 + \frac{0.05}{2}[1.0538 + 1.11549475] \\
y_2^{(2)} &= 1.10553236875
\end{aligned}$$

Next, we shall use Euler's Modified Formula with $y_2^{(2)}$ to find $y_2^{(3)}$

$$\begin{aligned}
y_2^{(3)} &= y_1 + \frac{h}{2}[f(x_1, y_1)]f(x_2, y_2^{(2)} \\
&= 1.0513 + \frac{0.05}{2}[f(0.05, 1.0513) + f(0.1, 1.10553236875)] \\
&= 1.0513 + \frac{0.05}{2} + [0.05^2 + 1.0513 + 0.1^2 + 1.10553236875] \\
&= 1.0513 + \frac{0.05}{2}[1.0538 + 1.11553236875] \\
y_2^{(3)} &= 1.10553330921875
\end{aligned}$$

Therefore,

$$\boxed{\mathbf{y_2 = 1.1055}}$$

23. Given that $\frac{dy}{dx} = x^2 + y,\ y(0) = 1$, determine $y(0.02)$ using Euler's modified method, correct upto five decimal places.

Given differential equation is

$$\frac{dy}{dx} = x^2 + y \text{ where } y(0) = 1$$

Now, to find $y(0.02)$ let us take $x = 0.02$.Taking h = 0.01, we get

$x_0 = 0.,\ x_1 = 0.01,\ x_2 = 0.02,$
Now, we shall find $y_0,\ y_1,\ y_2,\ ,$ using Euler's Modified method.

Given differential equation is

$$\frac{dy}{dx} = x^2 + y \text{ where } y(0) = 1$$

Now, to find $y(0.02)$ let us take $x = 0.02$.Taking h = 0.01, we get

$x_0 = 0.,\ x_1 = 0.01,\ x_2 = 0.02,$
Now, we shall find $y_0,\ y_1,\ y_2,\ ,$ using Euler's Modified method.

Finding $\mathbf{y(x_1) = y(0.01)}$

We shall find $y_1^{(0)}$ using Euler's Method.

$$\begin{aligned} y_1^{(0)} &= y_0 + hf(x_0, y_0) \\ &= 1 + 0.01f(0,1) \\ &= 1 + 0.01[0^2 + 1] \\ &= 1 + 0.01[1] \\ y_1^{(0)} &= 1.01 \end{aligned}$$

Next, we shall use Euler's Modified Formula with $y_1^{(0)}$ to find $y_1^{(1)}$

$$\begin{aligned} y_1^{(1)} &= y_0 + \frac{h}{2}[f(x_0, y_0)]f(x_1, y_1^{(0)} \\ &= 1 + \frac{0.01}{2}[f(0,1) + f(0.01, 1.01)] \\ &= 1 + \frac{0.01}{2} + [0^2 + 1 + 0.01^2 + 1.01] \\ &= 1 + \frac{0.01}{2}[1 + 1.0101] \\ y_1^{(1)} &= 1.0100505 \end{aligned}$$

Next, we shall use Euler's Modified Formula with $y_1^{(1)}$ to find $y_1^{(2)}$

$$\begin{aligned}
y_1^{(2)} &= y_0 + \frac{h}{2}[f(x_0, y_0)]f(x_1, y_1^{(1)} \\
&= 1 + \frac{0.01}{2}[f(0,1) + f(0.01, 1.0100505)] \\
&= 1 + \frac{0.01}{2} + [0^2 + 1 + 0.01^2 + 1.0100505] \\
&= 1 + \frac{0.01}{2}[1 + 1.0101505] \\
y_1^{(2)} &= 1.0100507525
\end{aligned}$$

Therefore,

$$\boxed{\mathbf{y_1 = 1.01005}}$$

Finding $\mathbf{y(x_2) = y(0.02)}$

We shall find $y_2^{(0)}$ using Euler's Method.

$$\begin{aligned}
y_2^{(0)} &= y_1 + hf(x_1, y_1) \\
&= 1.01005 + 0.01f(0.01, 1.01005) \\
&= 1.01005 + 0.01[0.01^2 + 1.01005] \\
&= 1.01005 + 0.01[1.01015] \\
y_2^{(0)} &= 1.0201515
\end{aligned}$$

Next, we shall use Euler's Modified Formula with $y_2^{(0)}$ to find $y_2^{(1)}$

$$\begin{aligned}
y_2^{(1)} &= y_1 + \frac{h}{2}[f(x_1, y_1)]f(x_2, y_2^{(0)} \\
&= 1.01005 + \frac{0.01}{2}[f(0.01, 1.01005) + f(0.02, 1.0201515)] \\
&= 1.01005 + \frac{0.01}{2} + [0.01^2 + 1.01005 + 0.02^2 + 1.0201515] \\
&= 1.01005 + \frac{0.01}{2}[1.01015 + 1.0205515] \\
y_2^{(1)} &= 1.0202035075
\end{aligned}$$

Next, we shall use Euler's Modified Formula with $y_2^{(1)}$ to find $y_2^{(2)}$

$$\begin{aligned}
y_2^{(2)} &= y_1 + \frac{h}{2}[f(x_1, y_1)]f(x_2, y_2^{(1)} \\
&= 1.01005 + \frac{0.01}{2}[f(0.01, 1.01005) + f(0.02, 1.0202035075)] \\
&= 1.01005 + \frac{0.01}{2} + [0.01^2 + 1.01005 + 0.02^2 + 1.0202035075] \\
&= 1.01005 + \frac{0.01}{2}[1.01015 + 1.0206035075] \\
y_2^{(2)} &= 1.0202037675375
\end{aligned}$$

Therefore,

$$\boxed{\mathbf{y_2 = 1.0202}}$$

Hence,

$$\boxed{\mathbf{y(0.02) = 1.0202}}$$

24. Runge-Kutta method of Second Order.

Runge-Kutta method of Second Order:

Let $\frac{dy}{dx} = f(x, y)$ be a differential equation with the initial condition $y(x_0) = y_0$. To solve the equation using Runge-Kutta method of Second Order, for $y(x_n)$ corresponding to some given value of $x = x_0 + nh$ where h is uniform length for each subintervals $[x_{i-1}, x_i]$, first we evaluate $y(x_1) = y_1$ using the formula.

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

where,

$$k_1 = hf(x_0, y_0) \quad \text{and} \quad k_2 = hf(x_0 + h, y_0 + k_1)$$

Similarly, we evaluate $y(x_2) = y_2$ using the formula.

$$y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$$

where,

$$k_1 = hf(x_1, y_1) \quad \text{and} \quad k_2 = hf(x_1 + h, y_1 + k_1)$$

Continuing similarly, finally we evaluate $y(x_n$ using the formula

$$y_n = y_{n-1} + \frac{1}{2}(k_1 + k_2)$$

where,

$$k_1 = hf(x_{n-1}, y_{n-1}) \quad \text{and} \quad k_2 = hf(x_{n-1} + h, y_{n-1} + k_1)$$

25. **Given that $\frac{dy}{dx} = y - x,\ y(0) = 2$, determine $y(0.1)$ and $y(0.2)$ using Runge-Kutta method, correct upto four decimal places**

Here we wish to solve $\frac{dy}{dx} = y - x$ subject to $y(0) = 2$.

We have $x_0 = 0,\ x_1 = x_0 + h = 0.1$, and $x_2 = x_0 + 2h = 0.2$.

Finding $y(0.1)$

On $[x_0, x_1]$, we have

$$k_1 = hf(x_0, y_0) = 0.1f(0, 2) = 0.1(2 - 0) = 0.2$$

Therefore,

$$k_2 = hf(x_0 + h, y_0 + k_1) = 0.1f(0 + 0.1, 2 + 0.2) = 0.1f(0.1, 2.2) = 0.1(2.2 - 0.1) = 0.21$$

Therefore,

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2) = 2 + \frac{1}{2}(0.2 + 0.21) = 2.2050$$

Thus, we get

$$y(0.1) = 2.2050$$

Finding $y(0.2)$

On $[x_1, x_2]$, we have

$$k_1 = hf(x_1, y_1) = 0.1f(0.1, 2.205) = 0.1(2.105) = 0.2105$$

Therefore,

$$k_2 = hf(x_1+h, y_1+k_1) = 0.1f(0.1+0.1, 2.205+0.2105) = 0.1f(0.2, 2.4155) = 0.1(2.4155-0.2) = 0.22155$$

Therefore,

$$y_2 = y_1 + \frac{1}{2}(k_1 + k_2) = 2.205 + \frac{1}{2}(0.2105 + 0.22155) = 2.4210$$

Thus, we get

$$y(0.2) = 2.4210$$