According to the New Syllabus of Sardar Patel University for B.Sc.

Semester-I

CALCULUS

Subject Code : US01CMTH21(T)

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Preface ...

This book on **Calculus** has been specially written to meet the requirements of the **B.Sc. Semester-I Subject Code : US01CMTH21(T)** students of **Sardar Patel University**.

The aim of the book is to present the subject in such a way that an average student may find no difficulty in understanding it. Each topic in the book is treated in an easy, clear style and at the same time the treatment is rigorous. The language is simple and easily understandable.

Each unit of the book contains complete theory and a fairly large number of solved examples. The number of unsolved examples is also numerous and they are well graded. Most of them have been selected from the standard books available on the subject.

The authors does not claim any originality. All available standard books on the subject have been freely consulted during the preparation of this book.

Any suggestion for the improvement of the book will be gratefully received.

The authors are very thankful to the publishers for their full cooperation in bringing out the book in the present nice form.

– The Authors

SARDAR PATEL UNIVERSITY

Syllabus for B.Sc. (MATHEMATICS) SEMESTER-I

US01CMTH21(T)

Unit-1 : Hyperbolic Functions, Successive Derivative, Higher Order Derivatives, n^{th} Derivatives of Standard Form, Leibnitz's Theorem and its Applications, L'Hospital Rule. Technics of Sketching Conics, Reflection Properties of Conics, Rotation of Axes and Second Degree Equations, Classification into Conics using Discriminant.

Unit-2 : Curve Tracing in Cartesian coordinates, Parametric Equations, Tracing of Parametric Curves, Polar coordinates, Curve Tracing in Polar coordinates, Polar Equation of Conics.

Unit-3 : Reduction Formulae for Integration of $\sin^n x$, $\cos^p x \cos^q x$, $\tan^n x$, $\cot^n x$, $\operatorname{cosec}^n x$, Volumes by Slicing, Disks and Washers Methods, Volumes by Cylindrical Shells, Arc length, Arc length of Parametric Curves and Polar Curves, Derivation of Intrinsic Equations of a Curve, Area of Surface of Revolution.

Unit-4 : Curvature, Radius of Curvature for Cartesian, Parametric and Polar Equations, Length of Arc as a Function.
Limit and Continuity of a Functions of Two Variables, Neighbourhood of a Point, Partial Derivatives, Euler's Theorem on Homogeneous Functions of Two and Three Variables, Theorem on Total Differentials, Differentiation of Composite and Implicit Functions.
Introduction to Vector Functions, Limits and Continuity of Vector Functions, Differentiation and Integration of Vector Functions.

Recommended Texts :

1. Shanti Narayan, Differential Calculus, 14th Edition, S. Chand and Company Ltd., Ne Delhi 1996.
   Chapter : 5, 6 (6.6 Only), 9, 10, 12 (12.1, 12.2, 12.3 Only)

2. Shanti Narayan, Integral Calculus, 14th Edition, S Chand and Company Ltd., New Delhi 1996.
   Chapter : 4 (Except 4.7, 4.8, 4.9), 8 (Except 8.5)
3. Maurice D. Weir, Joel Hass, Frank R. Giordano, Thoma's Calculus, Pearson Education, Delhi.
   Chapter : 6 (Only 6.1, 6.2, 6.5), 10 (10.1, 10.2, 10.3), 13 (Only 13.1)
4. H. M. Vasavada, Analytical Geometry of Two and Three Dimensions, 1992.
   Chapter : 2 (Only 11, 12, 13), 3, 4 (1 to 5)

Reference Texts :

1. Louis Leithod, The Calculus with Analytic Geometry, Harper-Collins Pub.
2. G.B. Thomas and R.L. Finney, Calculus, 9th Ed., Pearson, Education, Delhi, 2005.
3. M.J. Strauss, G.L. Bradley and K.J. Smith, Calculus, 3rd Ed., John Wiley and Sons (Asia) P. Ltd., Singapore, 2002.
4. H. Anton, I Bivens, S. Devis, Calculus, 7th Ed., John Wiley and sons (Asia) Pt. Ltd., Singapore, 2002.
5. D.J. Karia, N. Y Patel, B.P. Patel, M.L. Patel, Introduction to Calculus and Differential Equations, Roopal Prakashan, Vallabh Vidyanagar.
6. S.K. Patel, B.P. Patel, H.R. Kataria, B.L. Ghodadra, Calculus and Matrix Algebra, University Grant nirman board, Ahmedard-6.
7. B.S. Grewal, Higher Engineering Mathematics, Thirty-fifth edition, Khanna Publ.

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SARDAR PATEL UNIVERSITY

Syllabus for B.Sc. (MATHEMATICS) SEMESTER-I

US01CMTH22(P)

PROBLEMS AND EXERCISES IN CALCULUS

List of Practical :

1. Hyperbolic Functions, Successive Derivative, Higher Order Derivatives : n^{th} Derivatives of Standard Form, Leibnitz's Theorem and its Applications.
2. L'Hospital Rule, Technics of Sketching Conics, Reflection Properties of Conics, Rotation of axes and Second Degree Equations, Classification into Conics using Discriminant.
3. Curve Tracing in Cartesian coordinates, Parametric Equations, Tracing of Parametric Curves.
4. Polar coordinates, Curve Tracing in Polar coordinates, Polar Equation of Conics.
5. Reduction Formulae for Integration of $\sin^n x$, $\cos^n x$ $\sin^p x$, $\cos^q x$, $\tan^n x$, $\cot^n x$, $\sec^n x$, $\operatorname{cosec}^n x$, Volumes by Slicing, Disks and Washers Methods, Volumes by Cylindrical Shells.
6. Arc Length, Arc Length of Parametric Curves and Polar Curves, Derivation of Intrinsic Equations of a Curve, Area of Surface of Revolution.
7. Curvature, Radius of Curvature For Cartesian, Parametric and Polar Equations, Length of Arc as a Function.
8. Limit and Continuity of a Functions of Two Variables, Neighbourhood of a Point, Partial Derivatives, Euler's Theorem on Homogeneous Functions of Two and Three Variables.
9. Theorem on Total Differentials, Differentiation of Composite and Implicit Functions.
10. Introduction to Vector Functions, Limits and Continuity of Vector Functions, Differentiation and Integration of Vector Functions.

Note :

1. Problem solving skill in mathematics is an important aspect in the teaching of mathematics.
2. There would be a batch of problem solving session will be of four hours per week and, they will be conducted in batches of students of size 20 to 25 per batch.
3. The candidate shall have to produce at the time practical Examination the record of their prescribed Laboratory work, certified by the Head of the Department.

Recommended Texts :

1. Shanti Narayan, Differential Calculus, 14th Edition, S. Chand and Company Ltd., New Delhi 1996.
2. Shanti Narayan, Integral Calculus, 14th Edition, S. Chand and Company Ltd., New Delhi 1996.
3. B.S. Grewal, Higher Engineering Mathematics, 36th edition, Khanna Publ.
4. Dr. D.J. Karia, N.Y. Patel, B.P. Patel, M.L. Patel, Introduction to Calculus and Differential Equations, Roopal Prakashan, Vallabh Vidyanagar.
5. S.K. Patel, B.P. Patel, H.R. Kataria, B.L. Ghodadra, Calculus and Matrix Algebra, University Grant nirman board, Ahmedabad-6.
6. Louis Leithod, The Calculus with Analytic Geometry, Haper-Collins Pub.
7. G.B. Thomas and R.L. Finney, Calculus, 9th Ed., Pearson Education, Delhi, 2005.
8. M.J. Straus, G.L. Bradley and K.J. Smith, Calculus, 3rd Ed., John Wiley and Sons (Asia) P. Ltd., Singapore, 2002.
9. H. Anton, I. Bivens, S. Devis, Calculus, 7th Ed., John Wiley and sons (Asia) Pt. Ltd., Singapore, 2002.
10. Maurice D. Weir, Joel Hass, Frank R. Giordano, Thomas' Calculus, Pearson Education, Delhi.
11. H.M. Vasavada, Analytical Geometry of Two and Three Dimensions, 1992.

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Contents ...

TO THE READERS

Author and publisher would welcome suggestions towards future edition of this book or the pointing out of any misprint or obscurity. Please write to The Technical Editor, ATUL PRAKASHAN, Under Farnandis Bridge, Gandhi Road, Ahmedabad-1.

UNIT 1

Hyperbolic Functions, Higher Order Derivatives, Indeterminate Forms, Conic Sections

HYPERBOLIC FUNCTIONS

It is found useful to define sinh x, cosh x, tanh x, coth x, sech x, and cosech x as follows :

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \text{cosech}\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$\text{sech}\, x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

It may be seen that sinh$x$, tanh$x$, sech$x$ are defined $\forall x \in$ R and coth x, cosech x for all non-zero values of R.

The reader may prove directly that the following results hold good $\forall x \in$ R.

$\sinh(-x) = -\sinh x$, $\sinh(-x) = \cosh x$,

$\cosh^2 x - \sinh^2 x = 1$, $\cosh^2 x + \sinh^2 x = \cosh 2x$

$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$,

$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$.

It will be seen that sinh$x$, cosh$x$, etc. have properties analogous to those of sin$x$, cos$x$.

DERIVATIVES OF HYPERBOLIC FUNCTIONS

Example-1 : Obtain following :

1. Derivative of $y = \sinh x$; $x \in$ R

We have

$$y = \sinh x$$

$$= \frac{e^x - e^{-x}}{2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{2}(e^x - e^{-x})\right]$$

$$= \frac{1}{2}(e^x + e^{-x}) = \cosh x$$

$$\Rightarrow \quad \frac{d(\sinh x)}{dx} = \cosh x \ \forall x \in R.$$

2. Derivative of $y = \cosh x;\ x \in R$

We have

$$y = \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$\Rightarrow \quad \frac{d(\cosh x)}{dx} = \sinh x \ \forall x \in R.$$

3. Derivative of $y = \tanh x;\ x \in R$

$$y = \tanh x$$

$$= \frac{\sinh x}{\cosh x};\quad \cosh x \neq 0\ \forall x \in R.$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cosh x \cdot \frac{d(\sinh x)}{dx} - \sinh x \cdot \frac{d(\cosh x)}{dx}}{\cosh^2 x}$$

$$= \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{\cosh^2 x}$$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$$

$$= \frac{1}{\cosh^2 x}$$

$$= \operatorname{sech}^2 x;\ \forall x \in R.$$

$$\frac{d(\tanh x)}{dx} = \operatorname{sech}^2 x.$$

4. It may be similarly shown that

$$\frac{d(\coth x)}{dx} = -\operatorname{cosech}^2 x.$$

Proof is left to reader.

5. Derivative of $y = \operatorname{sech} x;\ x \in R$

We have

$$y = \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cosh x \cdot 0 - 1 \cdot \sinh x}{\cosh^2 x}$$

$$= \frac{-\sinh x}{\cosh^2 x} = -\tanh x \operatorname{sech} x;\ \forall x \in R.$$

$$\Rightarrow \quad \frac{d(\operatorname{sech} x)}{dx} = -\tanh x \operatorname{sech} x.$$

6. It may be similarly shown that

$$\frac{d(\operatorname{cosech} x)}{dx} = -\coth x \operatorname{cosech} x.$$

Proof is left to reader.

DERIVATIVES OF INVERSE HYPERBOLIC FUNCTIONS

Example-1 : Obtain following :

1. Derivative of $y = \sinh^{-1} x;\ x \in R$

Let

$$y = \sinh^{-1} x \text{ so that } x = \sinh y$$

$$\therefore \quad \frac{dx}{dy} = \cosh y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cosh y} = \pm \frac{1}{\sqrt{(1 + \sinh^2 y)}} = \pm \frac{1}{\sqrt{(1 + x^2)}}$$

where the sign of the radical is the same as that of cosh*y* which we know, is always positive

Hence,

$$\frac{d(\sinh^{-1} x)}{dx} = \frac{1}{\sqrt{(1 + x^2)}}.$$

2. Derivative of $\cosh^{-1}x$.

Let

$$y = \cosh^{-1}x \text{ so that } x = \cosh y$$

$$\Rightarrow \quad \frac{dx}{dy} = \sinh y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sinh y} = \pm\frac{1}{\sqrt{(\cosh^2 y - 1)}} = \pm\frac{1}{\sqrt{(x^2-1)}}$$

where the sign of the radical is the same as that of sinhy.

Now, $\cosh^{-1}x$ i.e., y is always positive so that sinhy is positive.

Hence,

$$\frac{d(\cosh^{-1}x)}{dx} = \frac{1}{\sqrt{(x^2-1)}}; \quad x > 1$$

3. Derivative of $\tanh^{-1}x$.

Let

$$y = \tanh^{-1}x \text{ so that } x = \tanh y$$

$$\Rightarrow \quad \frac{dx}{dy} = \text{sech}^2 y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\text{sech}^2 y} = \frac{1}{1-\tanh^2 y} = \frac{1}{1-x^2}$$

Thus,

$$\frac{d(\tanh^{-1}x)}{dx} = \frac{1}{1-x^2}; \quad |x| < 1$$

4. $\dfrac{d(\coth^{-1}x)}{dx} = \dfrac{-1}{x^2-1}$; $(|x| > 1)$.

Proof is left to the reader.

5. Derivative of $\text{sech}^{-1}x$.

Let

$$y = \text{sech}^{-1}x \text{ so that } x = \text{sech}\,y$$

$$\Rightarrow \quad \frac{dx}{dy} = -\text{sech}\,y \tanh y$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{\text{sech}\,y \tanh y}$$

$$= \pm\frac{-1}{\text{sech}\,y \cdot \sqrt{(1-\text{sech}^2 y)}} = \frac{-1}{x\sqrt{(1-x^2)}}$$

where the sign of the radical is the same as that tanh y.

But we know that $\text{sech}^{-1}x$, i.e., y is always positive, so that tanh y is always positive.

Hence,

$$\frac{d(\text{sech}^{-1}x)}{dx} = -\frac{1}{x\sqrt{(1-x^2)}}, \quad 0 < x < 1$$

6. Derivative of $\text{cosech}^{-1}x$.

Let

$$y = \text{cosech}^{-1}x \text{ so that } x = \text{cosech}\,y$$

$$\Rightarrow \quad \frac{dx}{dy} = -\text{cosech}\,y \coth y$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{\text{cosech}\,y \cdot \coth y}$$

$$= \pm\frac{-1}{\text{cosech}\,y \cdot \sqrt{(\text{cosech}^2 y + 1)}} = \pm\frac{-1}{x\sqrt{(x^2+1)}}$$

when the sign of the radical is the same as that coth y.

Now, y, and therefore coth y is positive or negative according as x is positive or negative.

$$\therefore \quad \frac{dy}{dx} = \frac{-1}{x\sqrt{(x^2+1)}} \text{ if } x > 0 \quad \text{and}$$

$$= \frac{-1}{-x\sqrt{(x^2+1)}} \text{ if } x < 0$$

Thus,

$$\frac{d(\text{cosech}^{-1}x)}{dx} = \frac{-1}{|x|\sqrt{(x^2+1)}}, \quad x \neq 0$$

Example-2 : Find the derivatives of

(1) $\log(\cosh x)$ (2) $e^{\sinh^2 x}$ (3) $\tan x \cdot \tanh x$

Example-3 : Find $\dfrac{dy}{dx}$

(1) $y = x^2 \sinh 2x$

$$\therefore \quad \frac{dy}{dx} = x^2 \cdot \cosh 2x \cdot 2 + \sinh 2x \cdot 2x$$

$$= 2x\,[x \cdot \cosh 2x + \sinh 2x]$$

(2) $y = \text{sech}^3(1-x^2)$

$$\therefore \quad \frac{dy}{dx} = 3\,\text{sech}^2(1-x^2)[-\text{sech}(1-x^2)\tanh(1-x^2)](0-2x)$$

$$= 6x\,\text{sech}^3(1-x^2)\tanh(1-x^2)$$

(3) $y = \tan^{-1}(\sinh x)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1+\sinh^2 x} \cosh x = \frac{\cosh x}{\cosh^2 x} = \operatorname{sech} x$$

(4) $y = \cosh^{-1}\frac{x}{2}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{\frac{x^2}{4}-1}} \cdot \frac{1}{2} = \frac{1}{\sqrt{(x^2-4)}}$$

(5) $y = \cosh(\sinh^{-1}x)$:

$$\therefore \quad \frac{dy}{dx} = \sinh(\sinh^{-1}x)\frac{1}{\sqrt{(x^2+1)}}$$

$$= \frac{x}{\sqrt{(1+x^2)}}$$

INTEGRATION OF HYPERBOLIC FUNCTIONS

The reader may prove following results directly :

1. $\int \sinh x\, dx = \cosh x + c$
2. $\int \cosh x\, dx = \sinh x + c$
3. $\int \operatorname{sech}^2 x\, dx = \tanh x + c$
4. $\int \operatorname{cosech}^2 x\, dx = -\coth x + c$
5. $\int \operatorname{sech} x \tanh x = \tanh x + c$
6. $\int \operatorname{cosech} x \coth x\, dx = -\operatorname{cosech} x + c$
7. $\int \frac{dx}{\sqrt{(x^2+a^2)}} = \sinh^{-1}\frac{x}{a} + c$
8. $\int \frac{dx}{\sqrt{(x^2-a^2)}} = \cosh^{-1}\frac{x}{a} + c$
9. $\int \frac{dx}{a^2-x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a} + c$
10. $\int \frac{dx}{x^2-a^2} = -\frac{1}{a}\coth^{-1}\frac{x}{a} + c$
11. $\int \tanh x\, dx = \ln|\cosh x| + c$

Example-1 : $\int \cosh^3 x \sinh x\, dx = \frac{1}{4}\cosh^4 x + c$

Example-2 : $\int x \operatorname{sech}^2 x\, dx = x\tanh x - \int \tanh x\, dx$

$= x\tanh x - \int \frac{\sinh x}{\cosh x} dx$ (Let $\cosh x = t$ then $\sinh x\, dx = dt$)

$= x\tanh x - \int \frac{1}{t} dt$

$= x\tanh x - \ln|t| + c$

$= x\tanh x - \ln|\cosh x| + c$

Example-3 : $\int \sinh^3 x\, dx = \int \frac{1}{4}(\sinh 3x - 3\sinh x)\, dx = \frac{1}{12}\cosh 3x - \frac{3}{4}\cos x + c$

Example-4 : $\int \frac{dx}{\sqrt{(x^2+2x+2)}} = \int \frac{dx}{\sqrt{(x+1)^2+1}} = \sinh^{-1}(x+1) + c$

Example-5 : $\int \frac{dx}{4x^2-9} = \frac{1}{4}\int \frac{dx}{x^2-\left(\frac{3}{2}\right)^2} = -\frac{1}{6}\coth^{-1}\left(\frac{2x}{3}\right) + c$

Example-6 : $\int \operatorname{sech} x\, dx = \int \frac{2dx}{e^x+e^{-x}} = 2\int \frac{e^x dx}{e^{2x}+1} = 2\tan^{-1}e^x + c$

Example-7 : $\int \operatorname{cosech} x\, dx = \int \frac{\operatorname{cosech} x(\coth x - \operatorname{cosech} x)}{(\coth x - \operatorname{cosech} x)} dx$

Let $\coth x - \operatorname{cosech} x = u$ then $(\coth x - \operatorname{cosech} x)\operatorname{cosech} x\, dx = du$

$= \int \frac{du}{u} = \ln|u| + c = \ln|\coth x - \operatorname{cosech} x| + c$

$= \ln\left|\frac{\cosh x - 1}{\sinh x}\right| + c$

$= \ln\left|\frac{2\sinh^2\frac{x}{2}}{2\sinh\frac{x}{2}\cosh\frac{x}{2}}\right| + c$

$= \ln\left|\tanh\frac{x}{2}\right| + c$

EXERCISE

Prove the following identities, using the definitions of the hyperbolic functions.

1. $\sinh 2x = 2\sinh x \cosh x$
2. $\cosh 2x = \cosh^2 x + \sinh^2 x$
3. $\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$
4. $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$
5. $\tanh^2 x + \operatorname{sech}^2 x = 1$
6. $\coth^2 x - \operatorname{cosech}^2 x = 1$
7. $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh x}$
8. $\tanh \frac{1}{2} x = \dfrac{\cosh x - 1}{\sinh x}$
9. $\operatorname{sech} x + \sinh x \tanh x = \cosh x$
10. $\sinh x + \operatorname{cosech} x = \cosh x \coth x$
11. $(\cosh x - \sinh x)^2 = \cosh 2x - \sinh 2x$
12. $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$
13. $\sinh 3x = 3\sinh x + 4\sinh^3 x$

Defferentiate each of the given functions in Problems 1 – 18.

1. $y = \sinh 3x$
2. $y = \ln \cosh 2x$
3. $y = \sinh^2 x$
4. $y = x^2 \sinh 2x$
5. $y = \ln \tanh^2 x$
6. $y = e^x \cosh x$
7. $s = \sinh t^2$
8. $s = \cosh \frac{1}{t}$
9. $s = \operatorname{sech}^3 (1 - x^2)$
10. $s = \tanh \sqrt{x}$
11. $s = \sqrt{\tanh t}$
12. $s = e^{\cosh t}$
13. $u = \cosh e^x$
14. $u = \cosh(\ln x)$
15. $u = \operatorname{sech}(\sin x)$
16. $u = \tan^{-1}(\sinh x)$
17. $y = \sin^{-1}(\tanh x)$
18. $y = \sin^{-1}(\sinh x)$
19. Derive the formula $\int \coth u \, du = \ln|\sinh u| + c$
20. Express $\operatorname{sech} u$ in exponential form and show that $\int \operatorname{sech} u \, du = 2\tan^{-1} e^u + c$
21. Show that $\int \operatorname{cosech} u \, du = \ln\left|\tanh \frac{u}{2}\right| + c$

Perform the following integrations.

22. $\int \sinh 3x \, dx$
23. $\int \operatorname{sech}^2 3x \, dx$
24. $\int x \operatorname{sech}^2 x \, dx$
25. $\int \cosh^3 x \sinh x \, dx$
26. $\int e^x \tanh e^x \, dx$
27. $\int \coth x \, dx$
28. $\int \sinh^3 x \, dx$
29. $\int \sinh^2 2x \, dx$
30. $\int \operatorname{sech}^3 2x \tanh 2x \, dx$
31. $\int \cosh^2 3x \, dx$
32. $\int \tanh^5 x \operatorname{sech}^2 x \, dx$
33. $\int \sin x \sinh x \, dx$

Differntiate the following functions :

1. $y = \sinh^{-1} 2x$
2. $y = \cosh^{-1} \frac{1}{2} x$
3. $y = \tanh^{-1} e^x$
4. $y = \sinh^{-1}(1 - x^2)$
5. $y = (\sinh^{-1} x)^2$
6. $y = \tanh^{-1} \frac{x}{a}$
7. $y = \cosh(\sinh^{-1} x)$
8. $y = \sinh^{-1}(\cos x)$
9. $y = \sinh^{-1}(\tan x)$
10. $y = \cosh^{-1}(\sec x)$

Evaluate the following integrals :

11. $\int \dfrac{dx}{\sqrt{x^2 + 9}}$
12. $\int \dfrac{dx}{\sqrt{9x^2 - 16}}$
13. $\int \dfrac{dx}{4x^2 - 9}$
14. $\int \dfrac{x \, dx}{4 - x^4}$
15. $\int \dfrac{dx}{x^2 - 2x}$
16. $\int_0^1 \dfrac{dx}{\sqrt{x^2 + 2x + 2}}$
17. $\int_2^3 \dfrac{dx}{x^2 - 1}$
18. $\int_2^4 \dfrac{dx}{\sqrt{x^2 - 1}}$
19. $\int_{-1}^1 \dfrac{dx}{9 - x^2}$
20. $\int_{-1}^1 \dfrac{dx}{\sqrt{x^2 + 4}}$

HIGHER ORDER DERIVATIVES

■ **Definition :**

Higher order derivative : Let $f : (a, b) \to R$ be differentiable on (a, b). If f' is differentiable, then we say that f is twice differentiable and f'' is called the second derivative of f. f'' is also denoted by $\frac{d^2f}{dx^2}$. Further, suppose the $(n-1)^{th}$ derivative $f^{(n-1)}$ of f exists. If $f^{(n-1)}$ is differentiable on (a, b), then its derivative is called the n^{th} derivative of f and is denoted by $f^{(n)}$ or $\frac{d^nf}{dx^n}$. If the function f is expressed in terms of $y = f(x)$, then the successive derivatives of y are denoted by $y_1, y_2, ..., y_n$ or $\frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^ny}{dx^n}$.

The following proposition describes the formulae for n^{th} order derivatives of some standard functions.

■ **Proposition-1 :**

Let $a, b, c \in R$. Then prove the following

(1) For an integer m if $y = (ax+b)^m$, then

$y_n = m(m-1)....(m-n+1)a^n(ax+b)^{m-n}$

(2) If $y = (ax+b)^m$, with $m \in N$, then $y_n = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$

SPU, September-2014

(3) If $y = (ax+b)^{-1}$, then $y_n = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$ **SPU, November-2011**

(4) For $y = \log(ax+b)$ $y_n = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$ **SPU, September-2014**

(5) For $y = a^{mx}$, $y_n = m^n(\log a)^n a^{mx}$

(6) For $y = e^{mx}$, $y_n = m^n e^{mx}$. **SPU, November-2011, 2010**

(7) For $y = \cos(ax+b)$, $y_n = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$

(8) For $y = \sin(ax+b)$, $y_n = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$

(9) For $y = e^{ax}\cos(bx+c)$, $y_n = r^n e^{ax}\cos(bx+c+n\varphi)$,

where $r = \sqrt{a^2+b^2}$, $\varphi = \tan^{-1}\left(\frac{b}{a}\right)$.

(10) For $y = e^{ax}\sin(bx+c)$, $y_n = r^n e^{ax}\sin(bx+c+n\varphi)$,

where $r = \sqrt{a^2+b^2}$, $\varphi = \tan^{-1}\left(\frac{b}{a}\right)$. **SPU, November-2015, 2013**

Proof :

(1) We prove this by Mathematical induction method :

For $n = 1$, $y_1 = ma(ax+b)^{m-1}$

Thus result is true for $n = 1$.

Assume that $y_n = m(m-1)(m-2)....(m-n+1)a^n(ax+b)^{m-n}$ for a fixed $n \in N$. Then

$$y_{n+1} = \frac{dy_n}{dx} = \frac{d}{dx}[m(m-1)...(m-n+1)a^n(ax+b)^{m-n}]$$

$$= m(m-1)....(m-n+1)a^n\frac{d}{dx}(ax+b)^{m-n})$$

$$= m(m-1)....(m-n+1)a^n(m-n)(ax+b)^{(m-n-1)}a$$

$$= m(m-1)....(m-(n+1)+1)a^{n+1}(ax+b)^{(m-(n+1))}$$

Thus result is true for $n + 1$.

Hence by Mathematical induction method we say that result is true for all $n \in N$.

(2) We prove this by Mathematical induction method :

For $n = 1$, $y_1 = ma(ax+b)^{m-1}$

Thus result is true for $n = 1$.

Assume that $y_n = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$ for a fixed $n \in N$. Then

$$y_{n+1} = \frac{dy_n}{dx} = \frac{d}{dx}\left(\frac{m!}{(m-n)!}a^n(ax+b)^{m-n}\right)$$

$$= \frac{m!}{(m-n)!}a^n\frac{d}{dx}((ax+b)^{m-n})$$

$$= \frac{m!}{(m-n)!}a^n(m-n)(ax+b)^{(m-n-1)}a$$

$$= \frac{m!}{(m-n-1)!} a^{n+1} (ax+b)^{(m-(n+1))}$$

$$= \frac{m!}{(m-(n+1))!} a^{n+1} (ax+b)^{(m-(n+1))}$$

Thus result is true for $n+1$.

Hence by Mathematical induction method we say that result is true for all $n \in$ N.

(3) Put $m = -1$ in above result (1), we get

$$y_n = -1(-1-1)\ldots(-1-n+1)a^n(ax+b)^{-1-n}$$

$$= \frac{(-1)^n(1.2.3.4\ldots n)a^n}{(ax+b)^{n+1}}$$

$$= \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$$

(4) We prove this by Mathematical induction method :

Here $y = \log(ax+b)$. For $n = 1$, $y_1 = \frac{1}{ax+b} a = \frac{(-1)^0(0)!a^1}{(ax+b)} = \frac{(-1)^{1-1}(1-1)!\,a^1}{(ax+b)^1}$

Thus result is true for $n = 1$.

Now assume that $y_n = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$. Then

$$y_{n+1} = \frac{dy_n}{dx} = \frac{d}{dx}\left(\frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}\right)$$

$$= (-1)^{n-1}(n-1)!a^n \frac{d}{dx}\left(\frac{1}{(ax+b)^n}\right)$$

$$= (-1)^{n-1}(n-1)!a^n \frac{d}{dx}(ax+b)^{-n}$$

$$= (-1)^{n-1}(n-1)!a^{n+1}(-n)(ax+b)^{-n-1}$$

$$= (-1)^n n! a^{n+1}(ax+b)^{-(n+1)}$$

$$= \frac{(-1)^n n! a^{n+1}}{(ax+b)^{(n+1)}}$$

Thus result is true for $n+1$.

Hence by Mathematical induction method we say that result is true for all $n \in$ N.

(5) $y = a^{mx}$.

Then $y_1 = a^{mx}(\log a)m = m(\log a)a^{mx}$.

Similarly we get, $y_2 = m^2(\log a)^2 a^{mx}$, $y_3 = m^3(\log a)^3 a^{mx}$. In general,

$$y_n = m^n(\log a)^n a^{mx}$$

(6) $y = e^{mx}$.

Then $y_1 = me^{mx}$.

Similarly we get, $y_2 = m^2e^{mx}$, $y_3 = m^3e^{mx}$.

In general, $y_u = m^n e^{mx}$.

(7) $y = \cos(ax+b)$.

Then, $y_1 = -a\sin(ax+b) = a\cos\left(ax+b+\frac{\pi}{2}\right)$

$$y_2 = -a^2\sin\left(ax+b+\frac{\pi}{2}\right) = a^2\cos\left(ax+b+\frac{2\pi}{2}\right)$$

$$y_3 = -a^3\sin\left(ax+b+\frac{2\pi}{2}\right) = a^3\cos\left(ax+b+\frac{3\pi}{2}\right)$$

....

$$y_n = -a^n\sin\left(ax+b+\frac{(n-1)\pi}{2}\right) = a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$$

(8) $y = \sin(ax+b)$.

Then, $y_1 = a\cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right)$

$$y_2 = a^2\cos\left(ax+b+\frac{\pi}{2}\right) = a^2\sin\left(ax+b+\frac{2\pi}{2}\right)$$

$$y_3 = a^3\cos\left(ax+b+\frac{2\pi}{2}\right) = a^3\sin\left(ax+b+\frac{3\pi}{2}\right)$$

....

$$y_n = a^n\cos\left(ax+b+\frac{(n-1)\pi}{2}\right) = a^n\sin\left(ax+b+\frac{n\pi}{2}\right)$$

(9) $y = e^{ax}\cos(bx+c)$.

Then $y_1 = ae^{ax}\cos(bx+c) - be^{ax}\sin(bx+c)$

$$= e^{ax}[a\cos(bx+c) - b\sin(bx+c)] \quad \ldots\ldots (1)$$

Let $a = r\cos\varphi$ and $b = r\sin\varphi$.

Then $r^2 = a^2 + b^2$ and $\dfrac{b}{a} = \tan\varphi$.

That is, $r = \sqrt{a^2+b^2}$, $\varphi = \tan^{-1}\left(\dfrac{b}{a}\right)$.

Now by (1),

$$y_1 = e^{ax}(r\cos\varphi\cos(bx+c) - r\sin\varphi\sin(bx+c))$$
$$= re^{ax}\cos(bx+c+\varphi)$$

Similarly, $y_2 = r^2e^{ax}\cos(bx+c+2\varphi)$

Continuing in this manner, in general we get,

$$y_n = r^n e^{ax}\cos(bx+c+n\varphi)$$

where $r = \sqrt{a^2+b^2}$, $\varphi = \tan^{-1}\left(\dfrac{b}{a}\right)$.

(10) $\boldsymbol{y = e^{ax}\sin(bx+c)}$.

Then $y_1 = ae^{ax}\sin(bx+c) + be^{ax}\cos(bx+c)$

$$= e^{ax}[a\sin(bx+c) + b\cos(bx+c)] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let $a = r\cos\varphi$ and $b = r\sin\varphi$.

Then $a^2+b^2 = r^2$ and $\dfrac{b}{a} = \tan\varphi$.

That is, $r = \sqrt{a^2+b^2}$, $\varphi = \tan^{-1}\left(\dfrac{b}{a}\right)$.

Now by (1),

$$y_1 = e^{ax}[r\cos\varphi\sin(bx+c) + r\sin\varphi\cos(bx+c)]$$
$$= re^{ax}\sin(bx+c+\varphi)$$

Similarly, $y_2 = r^2e^{ax}\sin(bx+c+2\varphi)$

Continuing in this manner, in general we get,

$$y_n = r^n e^{ax}\sin(bx+c+n\varphi)$$

where $r = \sqrt{a^2+b^2}$, $\varphi = \tan^{-1}\left(\dfrac{b}{a}\right)$.

■ **Example-1 :**

If y = cosmx − sinmx, then prove that $y_n = m^n\,(1-(-1)^n\sin 2mx)^{\frac{1}{2}}$

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Hence,
$$y_n = m^n\left[\cos\left(mx + n\tan^{-1}\left(\frac{m}{0}\right)\right) - \sin\left(mx + n\tan^{-1}\left(\frac{m}{0}\right)\right)\right]$$
$$= m^n\left[\cos(mx + n\tan^{-1}(\infty)) - \sin(mx + n\tan^{-1}(\infty))\right]$$
$$= m^n\left[\cos\left(mx + \frac{n\pi}{2}\right) - \sin\left(mx + \frac{n\pi}{2}\right)\right]$$
$$= m^n\left[\left[\cos\left(mx + \frac{n\pi}{2}\right) - \sin\left(mx + \frac{n\pi}{2}\right)\right]^2\right]^{\frac{1}{2}}$$
$$= m^n\left[1 - 2\cos\left(mx + \frac{n\pi}{2}\right)\sin\left(mx + \frac{n\pi}{2}\right)\right]^{\frac{1}{2}}$$
$$= m^n\,[1 - \sin(2mx + n\pi)]^{\frac{1}{2}}$$
$$= m^n\,[1 - \sin 2mx\cos n\pi - \cos 2mx\sin n\pi]^{\frac{1}{2}}$$
$$= m^n\,[1 - (-1)^n\sin 2mx]^{\frac{1}{2}}$$

■ **Example-2 :**

Find y_n for $y = e^{2x}\cos x\sin^2 2x$. SPU, April-2016, November-2010

Sol^n. :

Here,
$$y = e^{2x}\cos x\sin^2 2x$$
$$= e^{2x}\cos x\left(\frac{1-\cos 4x}{2}\right)$$
$$= \frac{1}{2}e^{2x}(\cos x - \cos 4x\cos x)$$
$$= \frac{1}{2}e^{2x}\left(\cos x - \frac{1}{2}(\cos 5x + \cos 3x)\right)$$
$$= \frac{1}{2}e^{2x}\cos x - \frac{1}{4}e^{2x}\cos 5x - \frac{1}{4}e^{2x}\cos 3x$$

$$\Rightarrow\quad y_n = \frac{5^{n/2}}{2}e^{2x}\cos\left(x + n\tan^{-1}\frac{1}{2}\right) - \frac{29^{n/2}}{2}e^{2x}\cos\left(5x + n\tan^{-1}\frac{5}{2}\right)$$
$$- \frac{13^{n/2}}{4}e^{2x}\cos\left(3x + n\tan^{-1}\frac{3}{2}\right)$$

LEIBNIZ'S RULE

■ **Theorem-1 :**

State and prove Leibniz's theorem :

SPU, April-2016, 2015; Dec. 2015; Sep. 2014; Nov. 2013, 2011, 2010; June-2012

Statement :

Let $u, v : E \to R$ be sufficiently many times differentiable functions. Then for any $n \in N$.

$$(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + ... + u v_n$$

Proof :

We prove this theorem by mathematical induction on n. Note that the result for $n = 1$ is $(uv)_1 = u_1 v + u v_1$, which is obviously true. Suppose that the result holds for $n = k$. That is,

$$(uv)_k = u_k v + {}^kC_1 u_{k-1} v_1 + {}^kC_2 u_{k-2} v_2 + ... + u v_k$$

Differentiating this we get :

$$(uv)_{k+1} = u_{k+1} v + u_k v_1 + {}^kC_1 (u_k v_1 + u_{k-1} v_2) + {}^kC_2 (u_{k-1} v_2 + u_{k-2} v_3) + ... + u_1 v_k + u v_{k+1}$$

$$= u_{k+1} v + (1 + {}^kC_1) u_k v_1 + ({}^kC_1 + {}^kC_2) u_{k-1} v_2 + ({}^kC_2 + {}^kC_3) u_{k-2} v_3 + ... + u v_{k+1}$$

$$= u_{k+1} v + {}^{k+1}C_1 u_k v_1 + {}^{k+1}C_2 u_{k-1} v_2 + {}^{k+1}C_3 u_{k-2} v_3 + ... + u v_{k+1}$$

as ${}^kC_{r-1} + {}^kC_r = {}^{k+1}C_r$. Thus the result is true for $n = k + 1$. Hence the result holds for all $n \in N$.

Note : Given a product of functions, usually it is a common practice to select the function as u whose n^{th} derivative is known us.

■ **Example-1 :**

If $y = x\log(x - 1)$, then find y_n.

Sol^n. :

Let $u = \log(x - 1)$, $v = x$, then by Leibniz's Theorem,

$$(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + ... + u v_n$$

Hence,

$$y_n = \frac{(-1)^{n-1}(n-1)!}{(x-1)^n} x + n \frac{(-1)^{n-2}(n-2)!}{(x-1)^{n-1}} 1 + 0 + ...$$

$$= \frac{(-1)^{n-2}(-1)(n-1)(n-2)!\,x}{(x-1)^n} + n \frac{(-1)^{n-2}(n-2)!}{(x-1)^{n-1}}$$

$$= \frac{(-1)^{n-2}(n-2)!}{(x-1)^n} [-x(n-1) + n(x-1)]$$

$$= \frac{(-1)^{n-2}(n-2)!}{(x-1)^n} (x - n)$$

■ **Example-2 :**

If $x = \cos\left(\frac{1}{m}\log y\right)$, then find $y_n(0)$. **SPU, April-2016, Dec. 2015**

Sol^n. :

We can write the given relation as

$$\cos^{-1} x = \frac{1}{m} \log y$$

$$\Rightarrow \quad m\cos^{-1} x = \log y$$

$$\Rightarrow \quad y = e^{m\cos^{-1} x} \qquad .... (3)$$

Then, $y_1 = e^{m\cos^{-1} x}\left(-\frac{m}{\sqrt{1-x^2}}\right)$

$$\Rightarrow \quad \sqrt{1-x^2}\, y_1 = -my \qquad .... (4)$$

Hence, $(1-x^2)y_1^2 = m^2 y^2$

By differentiating this we get,

$$2(1-x^2)y_1 y_2 - 2xy_1^2 = 2m^2 y y_1 \Rightarrow (1-x^2)\,y_2 - xy_1 = m^2 y \qquad .... (5)$$

Hence by Leibniz's Theorem, we get,

$$y_{n+2}(1-x^2) + {}^nC_1 y_{n+1}(-2x) - 2\,{}^nC_2 y_n - y_{n+1}x - {}^nC_1 y_n = m^2 y_n$$

$$\Rightarrow \quad (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n - m^2 y_n = 0$$

$$\Rightarrow \quad (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0 \qquad .... (6)$$

From (3) (4), (5) and (6) we have,

$$y(0) = e^{\frac{m\pi}{2}};\ y_1(0) = -me^{\frac{m\pi}{2}};\ y_2(0) = m^2 e^{\frac{m\pi}{2}};\ y_{n+2}(0) = (n^2+m^2)y_n(0) \qquad .... (7)$$

Putting $n = 1, 2, 3, ...$ in (7) we get,

$$y_3(0) = (1^2 + m^2)y_1(0) = -e^{\frac{m\pi}{2}} m(m^2+1);$$

$$y_4(0) = (2^2 + m^2)y_2(0) = e^{\frac{m\pi}{2}} m^2(m^2+2^2);$$

$$y_5(0) = (3^2 + m^2)y_3(0) = -e^{\frac{m\pi}{2}} m(m^2+3^2)$$

In general,

$$y_n(0) = \begin{cases} e^{\frac{m\pi}{2}} m^2(m^2+2^2)...(m^2+(n-2)^2) & \text{if } n \text{ is even;} \\ -e^{\frac{m\pi}{2}} m(m^2+1^2)...(m^2+(n-2)^2) & n \text{ odd; } n \neq 1 \end{cases}$$

■ **Example-3 :**

If $y=(x-\sqrt{4+x^2})^m$, then find $y_n(0)$. **SPU, April-2015; Sep. 2014; Nov. 2011**

Sol^n. :

Here $y=[x-\sqrt{4+x^2}]^m$ .... (8)

By differentiating with respect to x, we get,

$$y_1 = m(x-\sqrt{4+x^2})^{m-1}\left(1-\frac{2x}{2\sqrt{4+x^2}}\right)$$

$$= m(x-\sqrt{4+x^2})^{m-1}\left(\frac{\sqrt{4+x^2}-x}{\sqrt{4+x^2}}\right)$$

$$= -\frac{m(x-\sqrt{4+x^2})^m}{\sqrt{4+x^2}}$$

$$= -\frac{my}{\sqrt{4+x^2}}$$

$$\Rightarrow \sqrt{4+x^2}\,y_1 = -my. \quad .... (9)$$

Squaring both the sides,

$$(4+x^2)y_1^2 = m^2y^2$$

which, on differentiation given,

$$(4+x^2)2y_1y_2+2xy_1^2 = m^2 2yy_1 \Rightarrow (4+x^2)y_2+xy_1 = m^2y \quad .... (10)$$

By Leibniz's Theorem, we get,

$$y_{n+2}(4+x^2)+{}^nC_1y_{n+1}(2x)+2\,{}^nC_2y_n+y_{n+1}x+{}^nC_1y_n = m^2y_n$$

$$\Rightarrow (4+x^2)\,y_{n+2}+2nxy_{n+1}+n(n-1)y_n+xy_{n+1}+ny_n-m^2y_n=0$$

$$\Rightarrow (4+x^2)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0 \quad .... (11)$$

From (8), (9), (10) and (11) we have,

$$y(0)=(-2)^m;$$

$$y_1(0)=-m\frac{y(0)}{2}=-\frac{m}{2}(-2)^m;$$

$$y_2(0)=\frac{m^2y(0)}{4}=\frac{m^2}{4}(-2)^m;$$

$$y_{n+2}(0)=\frac{(m^2-n^2)y_n(0)}{4}$$

Putting $n = 1, 2, 3, ...$ in (12) to get,

$$y_3(0)=\frac{(m^2-1^2)y_1(0)}{4} = -\frac{m}{2}(-2)^m\frac{(m^2-1^2)}{4}$$

$$y_4(0)=\frac{(m^2-2^2)y_2(0)}{4} = \frac{m^2}{4}(-2)^m\frac{(m^2-2^2)}{4}$$

$$y_5(0)=\frac{(m^2-3^2)y_3(0)}{4} = -\frac{m}{2}(-2)^m\frac{(m^2-1^2)}{4}\frac{(m^2-3^2)}{4}$$

In general,
$$y_n(0)=\begin{cases}(-2)^m\dfrac{m^2}{4}\dfrac{(m^2-2^2)}{4}\cdots\dfrac{(m^2-(n-2)^2)}{4} & \text{if } n \text{ is even;}\\ -(-2)^m\dfrac{m}{2}\dfrac{(m^2-1^2)}{4}\cdots\dfrac{(m^2-(n-2)^2)}{4} & n \text{ odd; } n\neq 1\end{cases}$$

■ **Example-4 :**

Let $y=(x^2-2)^m$. Find the value of m such that
$(x^2-2)y_{n+2}+2xy_{n+1}-n(n+1)y_n=0$ **SPU, April-2015; November-2010**

Sol^n. :

Differentiating y with respect to x, we have,

$$y_1=m(x^2-2)^{m-1}2x$$

$$\Rightarrow (x^2-2)y_1=2mxy$$

$$\Rightarrow (x^2-2)y_2+2xy_1=2m(xy_1+y)$$

$$\Rightarrow (x^2-2)y_2+2(1-m)xy_1-2my=0$$

Hence, by applying Leibniz's Theorem, we get

$$(x^2-2)y_{n+2}+{}^nC_12xy_{n+1}+{}^nC_22y_n + 2(1-m)(xy_{n+1}+ny_n)-2my_n=0$$

$$\Rightarrow (x^2-2)y_{n+2}+2nxy_{n+1}+n(n-1)y_n + 2(1-m)(xy_{n+1}+ny_n)-2my_n=0$$

$$\Rightarrow (x^2-2)y_{n+2}+(n-m+1)2xy_{n+1}+(n^2-2mn+n-2m)y_n=0$$

$$\Rightarrow (x^2-2)y_{n+2}+(n-m+1)2xy_{n+1}+(n-2m)(n+1)y_n=0$$

Comparing the coefficients of the last equation with the given equation, we find that $m = n$.

■ **Example-5 :**

For $y = \sin(2x-3)+\cos(5x+1)$ find y_7. **SPU, April-2015**

Sol^n. :

$$y_7 = 2^7\sin\left(2x-3+\frac{7\pi}{2}\right)+5^7\cos\left(5x+1+\frac{7\pi}{2}\right)$$

$$= -2^7\cos(2x-3)+5^7\sin(5x+1)$$

■ **Example-6 :** **SPU, April-2015**

For $y = e^{3x} - \log(7x-5)$, find y_3.

Sol^n. :

$$y_3 = 3^3 e^{3x} - \frac{(-1)^{3-1}(3-1)!\,7^3}{(7x-5)^3}$$

$$= 3^3 e^{3x} - \frac{(-1)^2\, 2!\, 7^3}{(7x-5)^3}$$

$$= 3^3 e^{3x} - \frac{2 \cdot 7^3}{(7x-5)^3}$$

■ **Example-7 :** **SPU, September-2014**

If $y = e^{2x}\sin 5x$, then find y_4.

Sol^n. :

$$y_4 = \left[\sqrt{2^2+5^2}\right]^4 e^{2x}\sin\left(5x + 4\tan^{-1}\left(\frac{5}{2}\right)\right)$$

$$= (29)^2 e^{2x}\sin\left[5x + 4\tan^{-1}\left(\frac{5}{2}\right)\right]$$

■ **Example-8 :** **SPU, September-2014**

If $y = x^7$, find y_7.

Sol^n. :

Compare with $y = (ax+b)^m$ we get $a = 1$, $b = 0$, $m = 7$.

We know that $y_n = m(m-1)\,......(m-n+1)\,a^n\,(ax+b)^{m-n}$

$\therefore\ y_7 = 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 1^7\,(x)^{7-7}$

$= 7!\,x^0$

$y_7 = 7!.$

■ **Example-9 :** **SPU, November-2013**

If $y = \cos 3x$ then find y_4.

Sol^n. :

$$y_4' = 3^4\cos\left(3x + 4\,\frac{\pi}{2}\right)$$

$$= 3^4\cos(3x + 2\pi)$$

$$y_4 = 3^4\cos 3x$$

■ **Example-10 :** **SPU, November-2013**

Find y_n if $y = x\sin x$.

Sol^n. :

$y = x \cdot \sin x$

Let $u = \sin x$, $v = x$ by Leibnif's thorem,

$$y_n = u_n v + n_{C_1} u_{n-1} v_1 + u_{C_2} u_{n-2} v_2 + ... + u v_u$$

$$\therefore\ y_n = \sin\left(x + \frac{n\pi}{2}\right)x + n\cdot\sin\left(x + \frac{(n-1)\pi}{2}\right)$$

MULTIPLE CHOICE QUESTIONS

■ **Fill in the Blanks :**

1. If $y = (ax+b)^{-1}$ then y_n = ______.
   (a) $\frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$ (b) $\frac{(-1)^n n!\, a^n}{(ax+b)^n}$
   (c) $\frac{(-1)^n a^n}{(ax+b)^{n+1}}$ (d) $\frac{n!\, a^n}{(ax+b)^n}$
2. If $y = \log(ax+b)$ then y_n = ______.
   (a) $\frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$ (b) $\frac{(-1)^{n-1}(n-1)!\, a^n}{(ax+b)^n}$
   (c) $\frac{(-1)^n n!\, a^n}{(ax+b)^n}$ (d) $\frac{(-1)^{n-1} n!\, a^n}{(ax+b)^n}$
3. If $y = a^{mx}$ then y_n = ______.
   (a) $m^n \log a^n$ (b) $m^n \log a^n a^{mx}$
   (c) $m^n (\log a)^n a^{mx}$ (d) $m^n a^{mx}$
4. If $y = e^{mx}$ then y_n = ______.
   (a) e^{mx} (b) me^{mx}
   (c) $n^m e^{mx}$ (d) $m^n e^{mx}$
5. If $y = \cos(ax+b)$ then y_n = ______.
   (a) $a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$ (b) $a^n\cos(ax+b)$
   (c) $a^n\sin\left(ax+b+\frac{n\pi}{2}\right)$ (d) $\cos(ax+b+n\pi)$

6. If $y = \sin(ax+b)$ then $y_n =$ ______.

(a) $a^n \sin(ax+b)$
(b) $a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$
(c) $a^n \sin\left(ax+\frac{n\pi}{2}\right)$
(d) $\sin\left(ax+b+\frac{n\pi}{2}\right)$

7. If $y = (ax+b)^m$ then $y_2 =$ ______.

(a) $ma^2(ax+b)^{m-2}$
(b) $m(m-1)a^2(ax+b)^{m-1}$
(c) $m(m-1)a^2(ax+b)^{m-2}$
(d) $m^2(ax+b)^{m-2}$

8. If $y = (ax+b)^m$ then $y_2 =$ ______.

(a) $m!\,a^2(ax+b)^{m-2}$
(b) $\frac{m!}{2!}a^2(ax+b)^{m-2}$
(c) $m^2a^2(ax+b)^{m-2}$
(d) $\frac{m!}{(m-2)!}a^2(ax+b)^{m-2}$

9. If $y = e^{ax}\cos(bx+b)$ then $y_n =$ ______.

(a) $r^n e^{ax}\cos(bx+c+n\phi)$
(b) $r^{n/2}e^{ax}\cos(bx+c+n\phi)$
(c) $r^n e^{ax}\cos(bx+c)$
(d) $r^n e^{ax}\cos\left(bx+c+\frac{n\pi}{2}\right)$

10. $y = e^{ax}\cos bx$ then $y_n =$ ______.

(a) $r^n e^{ax}\cos(bx+c)$
(b) $r^n e^{ax}\cos(bx+n\phi)$
(c) $r^n e^{ax}\cos(bx+c+n\phi)$
(d) $e^{ax}\cos(bx+n\phi)$

11. ${}^kC_{(r-1)} + {}^kC_r =$ ______.

(a) ${}^{(k+1)}C_{r-1}$
(b) ${}^kC_{r+1}$
(c) ${}^{(k+1)}C_r$
(d) ${}^{2k}C_r$

12. If $y = e^{m\cos^{-1}x}$ then $\sqrt{1-x^2}\ y_1 =$ ______.

(a) my
(b) m^2y
(c) $-y$
(d) $-my$

13. If $y = e^{m\cos^{-1}x}$ then $y_1^2 =$ ______.

(a) $\frac{m^2y^2}{1-x^2}$
(b) $\frac{m^2y^2}{x^2-1}$
(c) $\frac{m^2y^2}{x^2+1}$
(d) $\frac{my^2}{1-x^2}$

14. If $y = [x-\sqrt{4+x^2}]^m$, then $\sqrt{4+x^2}\ y_1 + my =$ ______.

(a) 1
(b) 0
(c) y_1
(d) 2

15. If $x = \cos\left[\frac{1}{m}\log y\right]$, then $y(0) =$ ______.

(a) $e^{m\pi}$
(b) $-me^{\frac{m\pi}{2}}$
(c) $e^{\frac{m\pi}{2}}$
(d) 0

16. If $x = \cos\left[\frac{1}{m}\log y\right]$, then $y_1(0) =$ ______.

(a) $e^{\frac{m\pi}{2}}$
(b) $me^{\frac{m\pi}{2}}$
(c) $-me^{m\pi}$
(d) $-me^{\frac{m\pi}{2}}$

17. If $y = [x-\sqrt{4+x^2}]^m$, then $(4+x^2)\,y_1^2 =$ ______.

(a) m^2y^2
(b) my^2
(c) m^2y
(d) 0

18. If $y = [x-\sqrt{4+x^2}]^m$, then $y(0) =$ ______.

(a) 2^m
(b) $(-2)^m$
(c) $(-1)^m$
(d) 2^{-m}

19. If $y = [x-\sqrt{4+x^2}]^m$, then $y_1(0) =$ ______.

(a) $\frac{m}{2}(-2)^m$
(b) $(-1)^m\frac{m}{2}2^m$
(c) $-\frac{m}{2}(-2)^m$
(d) $-\frac{m}{2}2^m$

20. If $y = (x^2 - 2)^m$, then $(x^2 - 2)\, y_1$ = ____.

(a) $2my$ (b) mxy
(c) $2mx$ (d) $2mxy$

21. The n^{th} derivative of $y = \sin x$ is = ____.

(a) $\sin\left(2x + \frac{\pi}{2}\right)$ (b) $\cos\left(x + \frac{\pi}{2}\right)$
(c) $\sin\left(x + n\frac{\pi}{2}\right)$ (d) None of the above

22. The n^{th} derivative of $\cos(2x + 3)$ is = ____.

(a) $\sin\left(2x + 3 + \frac{\pi}{2}\right)$ (b) $\cos\left(2x + 3 + n\frac{\pi}{2}\right)$
(c) $\sin\left(2x + 3 + n\frac{\pi}{2}\right)$ (d) None of above

23. The 10^{th} derivative of a^{10x} is = ____.

(a) $10^{10}(\log 10)\, a^{10x}$ (b) $10^{10}(\log 10)^{10}\, a^{10x}$
(c) $10^{10} a^{10x}$ (d) None of the above

24. For the function u, v, Leibniz's rule gives n^{th} derivative of ______.

(a) $\frac{u}{v}$ (b) uv
(c) $\sqrt{uv}$ (d) $u + v$

25. n^{th} derivative of $y = \log x$ is ____.

(a) $\frac{(-1)^{n-1}(n-1)!}{x^n}$ (b) $\frac{(-1)^{n-1}n!}{x^n}$
(c) $\frac{(-1)^n(n-1)!}{x^n}$ (d) $\frac{(-1)^n(n-1)!}{x^{n-1}}$

26. The n^{th} derivative of a^{mx} is = ____.

(a) $a^m(\log a)a^{mx}$ (b) $a^m(\log a)^m a^{mx}$
(c) $m^n(\log a)^n a^{mx}$ (d) $m^n(\log a)a^{mx}$

27. The n^{th} derivative of the function $e^{3x}\cos 4x$ is = ____.

(a) $7^n e^{3x}\cos 4x$ (b) $5^n e^{3x}\cos\left(4x + n\tan^{-1}\frac{4}{3}\right)$
(c) $e^{3x}\cos\left(4x + n\frac{\pi}{2}\right)$ (d) None of the above

ANSWERS

1. (a),	2. (b),	3. (c),	4. (d),	5. (a),	6. (b),	7. (c),	8. (d),
9. (a),	10. (b),	11. (c),	12. (d),	13. (a),	14. (b),	15. (c),	16. (d),
17. (a),	18. (b),	19. (c),	20. (d),	21. (c),	22. (b),		
23. (b),	24. (b),	25. (a),	26. (c),	27. (b).			

SHORT QUESTIONS

1. Obtain the n^{th} derivative of the following functions :

(a) $\frac{1}{4x+3}$ (b) $\frac{1}{3x-5}$
(c) $(3x+4)^m$, $m \in N$ (d) $(4x-7)^m$, $m \in N$
(e) $\log(2x + 5)$ (f) 5^{4x}
(g) e^{7x} (h) $\cos(3x + 5)$
(i) $\sin(2x - 3)$ (j) $e^{3x}\sin(4x - 5)$
(k) $e^{3x}\cos(2x + 4)$

EXERCISE

1. Obtain the n^{th} derivative of the following functions :

(a) $e^{2x}\sin x\cos^2 2x$ (b) $\cos^4 x$
(c) $\sin^2 x\sin 2x$ (d) $\cos x\ \cos 2x\ \cos 3x$
(e) $e^{2x}\sin^3 x$ (f) $\sin 2x\ \sin 3x$
(g) $e^{3x} + \sin 2x\sin 3x$ (h) $\cos 2x\ \cos 3x\ \cos 4x$
(i) $\sin^5 x\cos^3 x$ (j) $(\tan^{-1}x)^2$
(k) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ (l) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$
(m) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ (n) $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$
(o) $\tan^{-1}\left(\frac{x}{a}\right)$

2. Find n^{th} derivative of the following functions :

(a) x^4e^{2x} (b) $x^3\cos x$

(c) $x^{n-1}\log x$ (d) $(2x+3)^2e^x$

(e) $(x+3)^3\sin 3x$ (f) $(x^2+9)^3\sin x$

(g) $x\sin x$ (h) $x^2\sin x$

(i) $x\cos x$ (j) $2\sin x\cos x$

3. If $y=\sin(m\sin^{-1}x)$, then prove that $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2-m^2)y_n=0$. Also find $y_n(0)$.

4. If $x = \sin mt$, $y = \cos mt$, then show that $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2-m^2)y_n=0$.

5. If $f(x) = \tan x$, then prove that $f^{(n)}(0) - {}^nC_2f^{(n-2)}(0) + {}^nC_4f^{(n-4)}(0) - .... = \sin\dfrac{n\pi}{2}$.

6. If $y=(x+\sqrt{1-x^2})^m$, then find $y_n(0)$.

7. If $x=\sin\left(\dfrac{\log y}{a}\right)$, then prove that $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2-a^2)y_n=0$. Also find $y_n(0)$.

8. If $y=(\sin^{-1}x)^2$, then prove that $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$.

9. If $y=e^{\tan^{-1}x}$, then prove that $(1+x^2)y_{n+2}+(2(n+1)x-1)y_{n+1}+n(n+1)y_n=0$.

10. If $y = \cos(\log x)$, then prove that, $x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0$.

11. If $\cos^{-1}\left(\dfrac{y}{b}\right)=\log\left(\dfrac{x}{n}\right)^n$, then prove that $x^2y_{n+2}+(2n+1)xy_{n+1}+2n^2y_n=0$.

12. If $y^{1/m}+y^{-1/m}=2x$, then prove that $(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$.

13. If $y = a\cos(\log x) + b\sin(\log x)$, then prove that $x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0$.

14. If $y=(\sinh^{-1}x)^2$, then prove that $(1+x^2)y_{n+2}+(2n+1)xy_{n+1}+n^2y_n=0$. Also find $y_n(0)$.

15. If $x=\cosh^{-1}\left(\dfrac{\log x}{m}\right)$, then prove that $(x^2+1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$

16. If $y=\log(x+\sqrt{1+x^2})$, then find $y_n(0)$.

17. If $y=[\log(x+\sqrt{1+x^2})]^2$, then find $y_n(0)$.

18. If $y=\cos(m\sin^{-1}x)$, then find $y_n(0)$.

INDETERMINATE FORMS

Introduction :

We have developed the algebra of limits and continuous functions in previous chapters. Let f, g be two suitably defined functions and $\Diamond$ stand for an algebraic operation. This means $\Diamond$ is one of the $+, -, \cdot, \div$. Then we have,

$$\lim_{x\to a}(f \Diamond g)(x) = \left(\lim_{x\to a} f(x)\right)\Diamond\left(\lim_{x\to a} g(x)\right). \quad ...(1)$$

Stated differently we say that the operation of taking limit is distributive over $\Diamond$. If we observe the results carefully, we can see that they are conclusive only with certain conditions.

In general, the validity of the above relation depends upon the situation.

Example-1 : Define $f : (0, \infty) \to R$ by $f(x)=\dfrac{1}{x}$ and let $g = f$.

Then $\lim_{x\to 0}(f(x) - g(x)) = \lim_{x\to 0}(f(x) - f(x)) = \lim_{x\to 0} = 0$.

However, $\lim_{x\to 0} f(x) - \lim_{x\to 0} g(x) = \infty - \infty$.

As we know, $\infty - \infty$ cannot be defined.

E in the following proposition contains some deleted neighbourhood of a. The proportion asserts, that limit may not exist in case of quotients of functions.

Proposition-1 :

Let f, g : E $\to$ R be two functions. Assume that

(1) $\lim_{x\to a} f(x)$ exists and is a nonzero real number.

(2) $\lim_{x\to a} g(x) = 0$.

Then prove that $\lim_{x\to a}\dfrac{f(x)}{g(x)}$ cannot be a finite number.

Proof :

Suppose, if possible, that $\lim_{x\to a}\dfrac{f(x)}{g(x)} = l \in$ R.

Then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) . \lim_{x\to a}\dfrac{f(x)}{g(x)} = 0 \cdot l = 0$, a contradiction.

Thus $\lim_{x\to a}\dfrac{f(x)}{g(x)}$ cannot be a finite number.

We devote this chapter to tackle such odd situations, where the previously developed classical tools fail. That is, we deal here with the so called indeterminate form. What is a indeterminate form ? There are various types of such forms. This forms occur while computing limits, there are some sort of conflicting forces at a certain point. A common example of this occurs when you have a ratio of two terms, each of which is getting arbitrarily close to zero, for example,

$$\lim_{x \to 0} \frac{\sin x}{x}$$

Here, as x goes to zero, the numerator is getting arbitrarily small, which would normally mean that the whole fraction is getting arbitrarily small; but in this case, the denominator is getting arbitrarily small, too, which would normally mean that the whole fraction is getting arbitrarily big – one force tries to make the fraction big, and the other tries to make it small.

The crux of the discussion is : "We shall try to make sense out of $\frac{0}{0}$." Who wins ? That's what L'Hospital's rule is designed to answer. The "rule" goes by the name of Guillaume Francois Antoine deo L'Hospital, Marquis de Sainte – Mesme (1651 – 1704), but it was actually discovered by John Bernoulli. It is by this rule that Marquis de L'Hopital is best known today but his greatest contribution to mathematics may have been as the author of the very first calculus textbook. Though later editions bore his name, the book first appeared anonymously in 1696, under somewhat dauting title "Analyse des infiniment petits pour l'intelligence des lignes courvbes", ("Analysis of the infinitely small, for the study of curves"). The book was based on (some say stolen from !) notes of John Bernoulli who had learned and molded the calculus after many years of correspondence with Gottfried Leibniz, the man who, together with Isaac Newton, is credited with the discovery of calculus.

■ $\frac{0}{0}$ form :

Recall that Proposition (1), does not conclude the existence of $\lim_{x \to a} \frac{f(x)}{g(x)}$, when $\lim_{x \to a} g(x) = 0$. Even if it exists, it may not be easier to compute the limit. The following theorem and results in the sequel show a computing scheme for such a limit, if it exists.

Theorem-1 :

(L'Hospital's rule for $\frac{0}{0}$ form). Let f and g be two differentiable functions and $a \in$ R. Suppose

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$$

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, when both the limits, $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ and $\lim_{x \to a} \frac{f(x)}{g(x)}$ exist.

Proof :

By applying the Mean Value Theorem to f and g, we have, $\theta_1, \theta_2 \in (0, 1)$ such that

$$\frac{f(a+h) - f(a)}{h} = f'(a + \theta_1 h); \quad \frac{g(a+h) - g(a)}{h} = g'(a + \theta_2 h).$$

Continuity of f and g at a gives $f(a) = \lim_{x \to a} f(x) = 0$, $g(a) = \lim_{x \to a} g(x) = 0$.

Hence,

$$\frac{f(a+h)}{g(a+h)} = \frac{f(a+h) - f(a)}{g(a+h) - g(a)} = \frac{f'(a+\theta_1 h)}{g'(a+\theta_2 h)}.$$

Now taking $h \to 0$ in this expression, we get,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{h \to 0} \frac{f(a+h)}{g(a+h)} = \lim_{h \to 0} \frac{f'(a+\theta_1 h)}{g'(a+\theta_2 h)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Theorem-2 :

(Generalized L'Hospitals' rule for $\frac{0}{0}$ form). Let f and g be sufficiently many times differentiable functions and $a \in$ R. Suppose

$$\lim_{x \to a} f^{(r)}(x) = \lim_{x \to a} g^{(r)}(x) = 0, \quad \text{for all } r = 0, 1, 2, \ldots, n - 1$$

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}$, when both the limits $\lim_{x \to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}$ and $\lim_{x \to a} \frac{f(x)}{g(x)}$ exist.

Remark :

L'Hospital's rule and Generalized L'Hospital's rule for $\frac{0}{0}$ form also hold when $a = \pm\infty$. This can be proved by substituting $x = \frac{1}{y}$ and then applying the corresponding L'Hospitals' rules for limits as $y \to 0$.

■ A word of warning :

The part of the hypothesis of L'Hospital's Rule that is most invitable is the existence of the limits, $\lim_{x \to a} \frac{f(x)}{g(x)}$ and $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ $\left(\lim_{x \to a} \frac{f^{(n)}(x)}{g^{(n)}(x)} \text{ for some } n \text{ in the generalized rule} \right)$.

One is led astray to believe that the nonexistence of the limit, $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ implies the nonexistence of the limit, $\lim_{x \to a} \frac{f(x)}{g(x)}$. This is untrue. There are examples makes it clear where the former limit exists but the latter does not. On the other hand there are examples where the former limit does not exist but the later does exist. It is this areas on, why we are using the phrase "when both the limits exist" (either finite or infinite) in the statements of L'Hospital's Rule (see Theorem-1). The conditions found usually in the statements of L'Hospital's Rule are :

(1) $g'(x)$ is non-zero in some deleted neighbourhood of a.

(2) $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$, where besides taking real values, L is allowed to take the values $\pm \infty$ also.

Many a times, it is easy to compute the limit, $\lim_{x \to a} \frac{f(x)}{g(x)}$ directly using the algebraic tools or some series representing the underlying functions instead of using L'Hospital's rule. In this connection, the following well-known series expansions of functions will become handy.

(1) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

(2) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

(3) $\tan x = x + \frac{x^3}{3} + \frac{2}{15} x^5 + \ldots$

(4) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$

(5) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$

(6) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

(7) $\tanh x = x - \frac{x^3}{3} + 2x^5 - \frac{17}{315} x^7 + \ldots, |x| < \frac{\pi}{2}$.

(8) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots, |x| < 1$.

(9) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots, |x| < 1$

(10) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots, |x| < 1$.

Here the base for the logarithm function is e. Let us also note that $\log 1 = 0$, $\log e = 1$. Also, we have the convention that $\log 0 = -\infty$, $\log \infty = \infty$.

Example-1 : Evaluate $\lim_{x \to 2} \frac{\sin(x^2 - 4)}{x - 2}$.

Solution :

As can be seen, the given limit is in $\frac{0}{0}$ form so by taking the derivatives,

$$\lim_{x \to 2} \frac{\sin(x^2-4)}{x-2} = \lim_{x \to 2} 2x \cos(x^2 - 4) = 4.$$

Example-2 : Evaluate $\lim_{x \to 0} \frac{e^x + \log(1-x) - 1}{\tan x - x}$.

Solution :

Evaluating the function at $x = 0$ we get the $\frac{0}{0}$ form. Hence by the L'Hospital's rule, we get

$$\lim_{x \to 0} \frac{e^x + \log(1-x) - 1}{\tan x - x}$$

$$= \lim_{x \to 0} \frac{e^x - \frac{1}{1-x}}{\sec^2 x - 1}$$

$$= \lim_{x \to 0} \frac{e^x(1-x) - 1}{(1-x)\tan^2 x}$$

$$= \lim_{x \to 0} \frac{e^x(1-x) - 1}{(1-x)x^2} \times \lim_{x \to 0} \frac{x^2}{\tan^2 x}$$

$$= \lim_{x \to 0} \frac{e^x(1-x) - 1}{x^2 - x^3} \quad \left(\text{again } \frac{0}{0} \text{ from}\right)$$

$$= \lim_{x \to 0} \frac{e^x - e^x x - e^x}{2x - 3x^2}$$

$$= \lim_{x \to 0} \frac{-e^x + e^x(1-x)}{2x - 3x^2} \quad \left(\text{again } \frac{0}{0} \text{ from}\right)$$

$$= \lim_{x \to 0} \frac{-e^x + e^x(1-x) + e^x(-1)}{2 - 6x}$$

$$= \lim_{x \to 0} \frac{-2e^x + e^x(1-x)}{2-6x}$$

$$= \frac{-2e^0 + e^0(1-0)}{2-6\cdot 0} = \frac{-1}{2}$$

As noted earlier, this limit can be computed using a series also.

$$\lim_{x \to 0} \frac{e^x + \log(1-x) - 1}{\tan x - x}$$

$$= \lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) + \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\right) - 1}{\left(x + \frac{x^3}{15}x^5 + \ldots\right) - x}$$

$$= \lim_{x \to 0} \frac{x^3\left(\frac{1}{6} - \frac{1}{3}\right) + x^4\left(\frac{1}{24} - \frac{1}{4}\right) + \ldots}{\frac{x^3}{3} + \frac{2}{15}x^5 + \ldots}$$

$$= \lim_{x \to 0} \frac{\frac{1}{6} - \frac{1}{3} + x\left(\frac{1}{24} - \frac{1}{4}\right) + \ldots}{\frac{1}{3} + \frac{2}{15}x^2 + \ldots}$$

$$= \frac{-\frac{1}{6}}{\frac{1}{3}} = -\frac{1}{2}$$

Example-3 : Find a, b, c so that $\lim_{x \to 0} \frac{ae^x - 2b\cos x + 3ce^{-x}}{x\sin x} = 2$.

Solution :

Since the denominator of the expression becomes zero at $x = 0$, in order to get a finite limit, numerator must also tend to zero as $x \to 0$. As a result,

$$\lim_{x \to 0} ae^x - 2b\cos x + 3ce^{-x} = 0 \quad \Rightarrow \quad a - 2b + 3c = 0. \qquad \ldots (1)$$

With this condition, our limit reduces to $\frac{0}{0}$ form. Hence by L'Hospital's rule, we get,

$$\lim_{x \to 0} \frac{ae^x - 2b\cos x + 3ce^{-x}}{x\sin x} = \lim_{x \to 0} \frac{ae^x + 2b\sin x - 3ce^{-x}}{\sin x + x\cos x}$$

where again the denominator is zero. So, to get the finite limit, we must have the numerator tending to 0 as $x \to 0$. As a result,

$$a - 3c = 0 \qquad \ldots (2)$$

Again with this condition, our limit reduces to $\frac{0}{0}$ form. Hence by L'Hospital's rule, we get,

$$2 = \lim_{x \to 0} \frac{ae^x + 2b\sin x - 3ce^{-x}}{\sin x + x\cos x} = \lim_{x \to 0} \frac{ae^x + 2b\cos x - 3ce^{-x}}{\cos x - x\sin x + \cos x}$$

Here the denominator tends to 2 as $x \to 0$. So, to get 2 as the answer, the numerator must tend to 4 as $x \to 0$. That is,

$$a + 2b + 3c = 4. \qquad \ldots (3)$$

On addition of (1) and (3) we get,

$$a + 3c = 2.$$

This, on addition with (2) gives,

$$a = 1,\ c = \frac{1}{3} \text{ and } b = 1.$$

■ $\frac{\infty}{\infty}$ **form :**

Theorem-1 : (L'Hospital's rule for $\frac{\infty}{\infty}$ form). Let f and g be two differentiable functions. Suppose

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty.$$

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, when both the limits, $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ and $\lim_{x \to a} \frac{f(x)}{g(x)}$ exist.

Proof :

First we rewrite the given expression to transform $\frac{\infty}{\infty}$ form into $\frac{0}{0}$ form. Thus,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{\frac{1}{g(x)}}{\frac{1}{f(x)}} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to a} \frac{\frac{-g'(x)}{(g(x))^2}}{\frac{-f'(x)}{(f(x))^2}}$$

$$= \lim_{x \to a} \frac{g'(x)}{f'(x)} \times \lim_{x \to a} \left(\frac{f(x)}{g(x)}\right)^2 \quad \ldots (1)$$

Now suppose $\lim_{x \to a} \frac{f(x)}{g(x)} = l$. The proof is now divided into different cases for different values of l.

Case-I : l is a nonzero real number. By (1), we get, $l = \lim_{x \to a} \frac{g'(x)}{f'(x)}$. Hence the result follow.

Case-II : $l = 0$. Hence we have $1 = \lim_{x \to a} \frac{f(x)}{g(x)} + 1$. That is,

$$1 = \lim_{x \to a} \frac{f(x) + g(x)}{g(x)},$$

which is in $\frac{\infty}{\infty}$ form with a finite nonzero limit. Hence as in the Case-I above,

$$1 = \lim_{x \to a} \frac{f'(x) + g'(x)}{g'(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} + 1.$$

Hence, $\lim_{x \to a} \frac{g'(x)}{f'(x)} = 0$, which proves the result.

Case-III : $l = \infty$. Hence $\lim_{x \to a} \frac{f(x)}{g(x)} = \infty$. Thus we have,

$$\lim_{x \to a} \frac{g(x)}{f(x)} = 0,$$

which is in $\frac{\infty}{\infty}$ form with limit 0. So, by Case-II above,

$$\lim_{x \to a} \frac{g(x)}{f(x)} = \lim_{x \to a} \frac{g'(x)}{f'(x)}$$

Hence, $\lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f(x)}{g(x)}$.

This completes the proof.

Remark :

The generalized L'Hospital's rule form $\frac{\infty}{\infty}$ can be obtain as in the $\frac{0}{0}$ form. Also, both the above theorem and its general form hold when we take $a = \pm\infty$.

Example-1 :

Can we apply L'Hospital's Rule to assert the existence of $\lim_{x \to \infty} \frac{x - \sin x}{x}$.

Solution :

The expression is in the for $\frac{\infty}{\infty}$.

On the other hand, we note that $\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} (1 - \cos x)$, which does not exist because we know that $1 - \cos x = 1$ for $x = \frac{(2k+1)\pi}{2}$ and $1 - \cos x = 0$ for $x = 2n\pi$. However, $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{x - \sin x}{x} = \lim_{x \to \infty} \left(1 - \frac{\sin x}{x}\right) = 1$. Note that $\lim_{x \to \infty} \frac{\sin x}{x} = 0$ because $\sin x$ is a bounded function and $\frac{1}{x} \to 0$, as $x \to \infty$. Thus the L'Hospital's rule fails here. This happens because $\lim_{x \to \infty} \frac{f'(x)}{g'(x)}$ does not make any sense in this case.

The following is a typically constructed example which asserts that the assumption that $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists, cannot be dropped from the hypothesis of L'Hospital's Rule.

Example-2 :

Can we apply L'Hospital's Rule to assert the existence of $\lim_{x \to \infty} \frac{x}{xe^{\sin x}}$?

Solution :

The answer is "no". In fact, as one can see, $e^{\sin x}$ goes on taking the values $e^{\pm 1}$ and 1 repeatedly as x approaches ∞. Thus $\lim_{x \to \infty} \frac{x}{xe^{\sin x}} = \lim_{x \to \infty} \frac{1}{e^{\sin x}}$ cannot exist. However, observe that the limit under discussion attains $\frac{\infty}{\infty}$ form. So, one is tempted to apply L'Hospital's Rule by taking $f(x) = x$ and $g(x) = xe^{\sin x}$. Now note that,

$$\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{1}{e^{\sin x} + x\cos x e^{\sin x}} = \lim_{x \to \infty} \frac{1}{e^{\sin x}(1 + x\cos x)} = 0,$$

because $e^{\sin x}$ is a bounded function and $|1 + x\cos x| \to \infty$, as $x \to \infty$. Thus in this situation $\lim\limits_{x \to \infty} \frac{f'(x)}{g'(x)}$ exists but $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)}$ does not exist. This shows that the assumption about the existence of the limit $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)}$ in L'Hospital's Rule is crucial and cannot be dropped.

Example-3 :

Evaluate $\lim\limits_{x \to 0} \frac{\log(\log(1-3x^2))}{\log(\log(\cos 2x))}$

Solution :

This expression takes $\frac{\infty}{\infty}$ form when $x \to 0$.

$$\lim_{x \to 0} \frac{\log(\log(1-3x^2))}{\log(\log(\cos 2x))} = \lim_{x \to 0} \frac{\frac{1}{\log(1-3x^2)}\left(\frac{-6x}{1-3x^2}\right)}{\frac{1}{\log(\cos 2x)}\left(\frac{-2\sin 2x}{\cos 2x}\right)}$$

$$= \lim_{x \to 0} \frac{1}{1-3x^2} \times \frac{\log(\cos 2x)}{\log(1-3x^2)} \times \frac{3}{2} \times \frac{2x}{\tan 2x}$$

$$= \frac{3}{2} \lim_{x \to 0} \frac{\log(\cos 2x)}{\log(1-3x^2)} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \frac{3}{2} \lim_{x \to 0} \frac{-2\tan 2x}{\frac{-6x}{1-3x^2}}$$

$$= \frac{3}{2} \lim_{x \to 0} \frac{2}{3} \frac{\tan 2x}{2x} \times (1-3x^2)$$

$$= 1.$$

Example-4 :

Evaluate $\lim\limits_{x \to 0} \frac{(\log 2x)^n}{x^{-m}}$, where $m, n \in$ N

Solution :

$$\lim_{x \to 0} \frac{(\log 2x)^n}{x^{-m}} \quad \left(\frac{\pm\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{\frac{n}{x}(\log 2x)^{n-1}}{-mx^{-m-1}}$$

$$= \lim_{x \to 0} \frac{n(\log 2x)^{n-1}}{-mx^{-m}} \quad \left(\frac{\pm\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{n(n-1)(\log 2x)^{n-2}}{(-m)^2 x^{-m}}$$

which is again in $\frac{\infty}{\infty}$ form. So, repeating this process, finally we get,

$$\lim_{x \to 0} \frac{n(n-1)(\log 2x)^{n-2}}{(-m)^2 x^{-m}}$$

$$= \lim_{x \to 0} \frac{n(n-1)\ldots.3\cdot 2\cdot 1}{(-m)^n x^{-m}}$$

$$= \lim_{x \to 0} \frac{n!}{(-m)^n} x^m = 0.$$

OTHER FORMS ($\infty - \infty$, $0 \times \infty$, 0^0, 1^∞ AND ∞^0)

Let f and g be two differentiable functions.

(1) Suppose $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = \infty$. Then $\lim\limits_{x \to a}(f(x) - g(x))$ is called the indeterminate form of $\infty - \infty$ type. In this case, we can write

$$\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} \frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)\,g(x)}},$$

which is in a $\frac{0}{0}$ form. We shall also see that it is convenient to reduce the limit expression to $\frac{\infty}{\infty}$ form by taking the reciprocals of numerator and denominator of the main fraction and then apply the L'Hoispital's rule.

(2) Suppose $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = \infty$. Then the $\lim\limits_{x \to a} f(x)\,g(x)$ is called the indeterminate form of $0 \times \infty$ type.

In this case, we can write,

$$\lim_{x\to a} f(x)\, g(x) = \lim_{x\to a} \frac{f(x)}{\frac{1}{g(x)}}, \qquad \left(\frac{0}{0} \text{ form}\right)$$

or $$\lim_{x\to a} f(x)\, g(x) = \lim_{x\to a} \frac{g(x)}{\frac{1}{f(x)}}, \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

(3) Let us now consider $l = \lim_{x\to a} f(x)^{g(x)}$. It may take different forms like 0^0, 1^∞ or ∞^0.

In such a situation, first we take log on both the sides to get $\log l = \lim_{x\to a} g(x) \log(f(x))$.

This form is now standard. After computations we take the exponent on both the sides to obtain l.

Example-1 :

Evaluate $\lim_{x\to 0}\left(\frac{1}{2x^2} - \frac{\cot^2 x}{2}\right)$.

Solution :

$$\lim_{x\to 0}\left(\frac{1}{2x^2} - \frac{\cot^2 x}{2}\right) \qquad (\infty - \infty \text{ form})$$

$$= \lim_{x\to 0}\left(\frac{1}{2x^2} - \frac{1}{2\tan^2 x}\right)$$

$$= \lim_{x\to 0} \frac{\tan^2 x - x^2}{2x^2 \tan^2 x}$$

$$= \lim_{x\to 0} \frac{\tan^2 x - x^2}{2x^4} \times \frac{x^2}{\tan^2 x}$$

$$= \lim_{x\to 0} \frac{\tan^2 x - x^2}{2x^4} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{2\tan x \sec^2 x - 2x}{8x^3} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\sec^4 x + 2\tan^2 x \sec^2 x - 1}{12x^2} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{4\sec^4 x \tan x + 4\tan x \sec^4 x + 4\tan^3 x \sec^2 x}{24x}$$

$$= \frac{1}{6} \lim_{x\to 0} \frac{\tan x}{x} (2\sec^4 x + \tan^2 x \sec^2 x)$$

$$= \frac{1}{6} \times 2$$

$$= \frac{1}{3}.$$

Algebraic Method :

$$\lim_{x\to 0}\left(\frac{1}{2x^2} - \frac{\cot^2 x}{2}\right)$$

$$= \lim_{x\to 0}\left(\frac{1}{2x^2} - \frac{1}{2\tan^2 x}\right)$$

$$= \lim_{x\to 0} \frac{\tan^2 x - x^2}{2x^2 \tan^2 x}$$

$$= \lim_{x\to 0} \frac{\tan^2 x - x^2}{2x^4} \times \frac{x^2}{\tan^2 x}$$

$$= \lim_{x\to 0} \frac{\tan^2 x - x^2}{2x^4}$$

$$= \lim_{x\to 0} \frac{\left[x + \frac{x^3}{3} + \dots\right]^2 - x^2}{2x^4}$$

$$= \lim_{x\to 0} \frac{\left[x^2 + 2\frac{x^4}{3} + \frac{x^6}{9} + \dots\right] - x^2}{2x^4}$$

$$= \lim_{x\to 0} \frac{x^4\left[\frac{2}{3} + 2\frac{x^4}{3} + \frac{x^6}{9} + \dots\right] - x^2}{2x^4}$$

$$= \frac{1}{2} \lim_{x\to 0}\left(\frac{2}{3} + \text{terms containing powers of } x\right)$$

$$= \frac{1}{3}.$$

Example-2 :

Evaluate $\lim\limits_{x \to a} (a - x) \tan\left(\dfrac{5\pi x}{2a}\right)$.

Solution :

$$\lim_{x \to a} (a - x) \tan\left(\frac{5\pi x}{2a}\right) \qquad (0 \cdot \infty \text{ form})$$

$$= \lim_{x \to a} \frac{a - x}{\cot\left(\dfrac{5\pi x}{2a}\right)} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to a} \frac{-1}{-\operatorname{cosec}^2\left(\dfrac{5\pi x}{2}\right)} \times \frac{2a}{5\pi}$$

$$= \frac{2a}{5\pi}.$$

Example-3 :

Evaluate $\lim\limits_{x \to a} \sin^{-1}\left(\sqrt{\dfrac{a - x}{3a + x}}\right) \operatorname{cosec}\left(\sqrt{a^2 - x^2}\right)$

Solution :

$$\lim_{x \to a} \sin^{-1}\left(\sqrt{\frac{a - x}{3a + x}}\right) \operatorname{cosec}\left(\sqrt{a^2 - x^2}\right)$$

$$= \lim_{x \to a} \frac{\sin^{-1}\left(\sqrt{\dfrac{a - x}{3a + x}}\right)}{\sin\left(\sqrt{a^2 - x^2}\right)}$$

$$= \lim_{x \to a} \frac{\sin^{-1}\left(\sqrt{\dfrac{a - x}{3a + x}}\right)}{\sqrt{\dfrac{a - x}{3a + x}}} \sqrt{\frac{a - x}{3a + x}}\, \frac{\sqrt{a^2 - x^2}}{\sin\sqrt{a^2 - x^2}}\, \frac{1}{\sqrt{a^2 - x^2}}$$

$$= \lim_{x \to a} \sqrt{\frac{a - x}{3a + x}}\, \frac{1}{\sqrt{a - x}\,\sqrt{a + x}}$$

$$= \lim_{x \to a} \frac{1}{\sqrt{3a + x}\,\sqrt{a + x}}$$

$$= \frac{1}{\sqrt{4a}\,\sqrt{2a}}$$

$$= \frac{1}{2\sqrt{2}\,a}.$$

Example-4 :

Evaluate $\lim\limits_{x \to 1} (4 - 4x^2)^{\frac{1}{\log(2 - 2x)}}$

Solution :

Let $y = \lim\limits_{x \to 1} (4 - 4x^2)^{\frac{1}{\log(2 - 2x)}}$ (0^0 form). Then

$$\log y = \lim_{x \to 1} \frac{1}{\log(2 - 2x)} \log(4 - 4x^2) \qquad (0 \cdot (-\infty) \text{ form})$$

$$= \lim_{x \to 1} \frac{\log(4 - 4x^2)}{\log(2 - 2x)} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to 1} \frac{\dfrac{-8x}{4 - 4x^2}}{\dfrac{-2}{2 - 2x}}$$

$$= \lim_{x \to 1} \frac{4x}{2 + 2x} = 1$$

Thus $\log y = 1$

$\Rightarrow \quad y = e^1 = e.$

Example-5 :

Evaluate $\lim\limits_{x \to 0} \left(\dfrac{\tan x}{x}\right)^{\frac{5}{3x^2}}$.

Solution :

Let $y = \lim\limits_{x \to 0} \left(\dfrac{\tan x}{x}\right)^{\frac{5}{3x^2}}$ (1^∞ form). Then

$$\log y = \lim_{x\to 0} \frac{5}{3x^2} \log\left(\frac{\tan x}{x}\right) \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{5}{6x}\left[\frac{x}{\tan x}\left(\frac{x\sec^2 x - \tan x}{x^2}\right)\right]$$

$$= \frac{5}{6}\lim_{x\to 0} \frac{x\sec^2 x - \tan x}{x^3} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \frac{5}{6}\lim_{x\to 0} \frac{2x\sec^2 x\tan x + \sec^2 x - \sec^2 x}{3x^2}$$

$$= \frac{5}{6}\lim_{x\to 0} \frac{2\sec^2 x}{3}\,\frac{\tan x}{x} = \frac{5}{9}$$

Thus $\log y = \frac{5}{9}$

$$\Rightarrow \quad y = e^{\frac{5}{9}}$$

Algebraic Method :

Let $y = \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{5}{3x^2}}$ (1^∞ form). Then

$$\log y = \lim_{x\to 0} \frac{5}{3x^2} \log\left(\frac{\tan x}{x}\right)$$

$$= \lim_{x\to 0} \frac{5}{3x^2} \log\left[\frac{x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots}{x}\right]$$

$$= \lim_{x\to 0} \frac{5}{3x^2} \log\left[1 + \frac{x^2}{3} + \frac{2}{15}x^4 + \ldots\right]$$

$$= \lim_{x\to 0} \frac{5}{3x^2} \log\left[1 + \left(\frac{x^2}{3} + \frac{2}{15}x^4 + \ldots\right)\right]$$

$$= \lim_{x\to 0} \frac{5}{3x^2} \left[\left(\frac{x^2}{3} + \frac{2}{15}x^4 + \ldots\right) - \frac{1}{2}\left(\frac{x^2}{3} + \frac{2}{15}x^4 + \ldots\right)^2 + \ldots\right]$$

$$= \lim_{x\to 0} \frac{5}{3x^2}\left[\frac{x^2}{3} + \text{terms containing higher powers of } x\right]$$

$$= \lim_{x\to 0} \frac{5}{3}\left[\frac{1}{3} + \text{terms containing powers of } x\right]$$

$$= \frac{5}{9}$$

$$\Rightarrow \quad y = e^{\frac{5}{9}}.$$

Example-6 :

Evaluate $\lim_{x\to 0} (\cot x)^{\sin 2x}$.

Solution :

Let $y = \lim_{x\to 0} (\cot x)^{\sin 2x}$ (∞^0 form). Then

$$\log y = \lim_{x\to 0} \sin 2x \log(\cot x)$$

$$= \lim_{x\to 0} \frac{\log(\cot x)}{\operatorname{cosec} 2x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\frac{-\operatorname{cosec}^2 x}{\cot x}}{-2\operatorname{cosec} 2x\cot 2x}$$

$$= \lim_{x\to 0} \frac{\operatorname{cosec}^2 x}{2\cot x\operatorname{cosec} 2x\cot 2x}$$

$$= \lim_{x\to 0} \frac{\sin 2x\tan x\tan 2x}{2\sin^2 x}$$

$$= \lim_{x\to 0} \tan 2x = 0$$

Thus $\log y = 0$

$$\Rightarrow \quad y = e^0 = 1.$$

The following example evaluates the limit at the point where the function is not defined. Thus actually, we need the function to be defined in a deleted neighbourhood of the point under consideration.

Example-7 :

Evaluate $\lim_{x\to\pi/2^-}\left(x-\frac{\pi}{2}\right)\tan x.$

Solution :

It is clear that $\left(x-\frac{\pi}{2}\right)\to 0$ and $\tan x\to\infty$ as $x\to\pi/2^-$. So this is $0\cdot\infty$ form. Using L'Hoispital's rule,

$$\lim_{x\to\pi/2^-}\left(x-\frac{\pi}{2}\right)\tan x=\lim_{x\to\pi/2^-}\frac{x-\frac{\pi}{2}}{\cot x}\qquad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to\pi/2^-}\frac{1}{(-\operatorname{cosec}^2 x)}=\lim_{x\to\pi/2^-}-\sin^2 x=-1.$$

EXERCISE

1. Evaluate the following :

(1) $\lim_{x\to0}\frac{x-\sin x}{\tan^3 x}$

(2) $\lim_{x\to0}\frac{\log(1-x^2)}{\log(\cos x)}$

(3) $\lim_{x\to0}\frac{e^x-e^{-x}-2\log(1+x)}{x\sin x}$

(4) $\lim_{x\to0}\frac{e^{\sin x}-1-x^2}{x^2+x\log(1-x)}$

(5) $\lim_{x\to1}\frac{2\sin\pi x+\pi(x^2-1)^2}{(x^2-1)^2}$

(6) $\lim_{x\to1}\frac{x\sqrt{3x-2x^4}-x^{6/5}}{1-x^{2/3}}$

(7) $\lim_{x\to0}\frac{e^x+e^{-x}+2\cos x-4}{x^4}$

(8) $\lim_{x\to0}\frac{e^x-1+\log(1+x)}{x^2}$

(9) $\lim_{x\to0}\frac{x-\tan x}{x^3}$

(10) $\lim_{x\to0}\frac{\log(1+x^3)}{\sin^3 x}$

(11) $\lim_{x\to0}\frac{\tan x\tan^{-1}x-x^2}{x^6}$

(12) $\lim_{x\to0}\frac{x-\sin^{-1}x}{x\sin^2 x}$

(13) $\lim_{x\to\pi/4}\frac{\sec^2 x-2\tan x}{1+\cos 4x}$

(14) $\lim_{x\to0}\frac{e^x+\sin x-1}{\log(1+x)}$

(15) $\lim_{x\to0}\frac{\sin x-x+\frac{x^3}{6}}{x^5}$

(16) $\lim_{x\to a}\frac{x^a-a^x}{x^x-a^a}$

(17) $\lim_{x\to0}\frac{\log_{\sin x}(\cos x)}{\log_{\sin\frac{x}{2}}\left(\cos\frac{x}{2}\right)}$

(18) $\lim_{x\to0}\frac{\log(\sin 2x)}{\log(\sin x)}$

(19) $\lim_{x\to0}\log_{\tan x}(\tan 2x)$

(20) $\lim_{x\to1/2}\frac{\tan(3\pi x)}{\sec(\pi x)}$

(21) $\lim_{x\to\infty}\frac{\log(1+e^{3x})}{x}$

(22) $\lim_{x\to\pi/2}\frac{\log\left(x-\frac{\pi}{2}\right)}{\tan x}$

(23) $\lim_{x\to a}\frac{\log(x-a)}{\log(e^x-e^a)}$

(24) $\lim_{x\to0}\frac{\log(\sin x)}{\cot x}$

(25) $\lim_{x\to\pi/2}\left(\tan x-\frac{2x\sec x}{\pi}\right)$

(26) $\lim_{x\to0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)$

(27) $\lim_{x\to0}\left(\frac{1}{x^2}-\cot^2 x\right)$

(28) $\lim_{x\to2}\left(\frac{1}{x-2}-\frac{1}{\log(x-1)}\right)$

(29) $\lim_{x\to1}\left(\frac{2}{x^2-1}-\frac{1}{x-1}\right)$

(30) $\lim_{x\to0}\left(\frac{\pi}{4x}-\frac{\pi}{2x(e^{\pi x}+1)}\right)$

(31) $\lim_{x\to\pi/2}\left(\sec x-\frac{1}{1-\sin x}\right)$

(32) $\lim_{x\to0}\left(\frac{1}{x}-\frac{\log(1+x)}{x^2}\right)$

(33) $\lim_{x\to0}\left(\operatorname{cosec}^3 x-\frac{1}{x^3}\right)$

(34) $\lim_{x\to\infty}(\cosh^{-1}x-\log x)$

(35) $\lim_{x\to1}\log(1-x)\cot\left(\frac{\pi x}{2}\right)$

(36) $\lim_{x\to0}\log\left(\frac{1+x}{1-x}\right)\cot x$

(37) $\lim_{x\to0}x\log x$

(38) $\lim_{x\to\infty}(\sqrt{x+1}-\sqrt{x})\log\frac{1}{x}$

(39) $\lim_{x\to a}\sqrt{\frac{a+x}{a-x}}\tan^{-1}(\sqrt{a^2-x^2})$

(40) $\lim_{x\to\infty}(a^{1/x}-1)x$

(41) $\lim_{x\to0}(\cos x)^{1/x^2}$

(42) $\lim_{x\to0}\frac{\cosh x+\log(1-x)-1+x}{x^2}$

(43) $\lim_{x\to0}\frac{(e^x-1)}{x^4\sin x}\left(\frac{3\sin x-\sin 3x}{\cos x-\cos 3x}\right)^4$

(44) $\lim_{x\to0}\frac{\sin 2x+2\sin^2 x-2\sin x}{\cos x-\cos^2 x}$

(45) $\lim_{x\to\infty}\frac{\log x}{x^n}$, where n is a constant

(46) $\lim_{x\to0}\left(\frac{\tan x}{x}\right)^{1/x^2}$

(47) $\lim_{x\to0}\left(\frac{\sinh x}{x}\right)^{1/x^2}$

(48) $\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

(49) $\lim_{x\to0}\left(\frac{a^x+b^x+c^x}{3}\right)^{1/x}$

(50) $\lim_{x\to0}(\operatorname{cosec}x)^{1/\log x}$

(51) $\lim_{x\to\pi/2}(\operatorname{cosec}x)^{\tan^2 x}$

(52) $\lim_{x\to0}(1+\tan x)^{\cot x}$

(53) $\lim_{x\to\infty}\left(\frac{1^{1/x}+2^{1/x}+3^{1/x}}{3}\right)^{3x}$

(54) $\lim_{x \to \infty} \left(\frac{x+1}{x-1}\right)^x$

(55) $\lim_{x \to 0} (\cos x)^{\cot^2 x}$

(56) $\lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\cot(x-a)}$

(57) $\lim_{x \to 1} (x-1)^{x-1}$

(58) $\lim_{x \to 0} \left(\frac{2(\cosh x - 1)}{x^2}\right)^{1/x^2}$

(59) $\lim_{x \to 0} (1+x)x^2 e^{1/x^2} \left[\frac{e^x - e^{-x}}{\log(1+x)}\right]$

(60) $\lim_{x \to \infty} \frac{x_n}{e^x}$, for a fixed positive integer n.

(61) $\lim_{x \to \infty} \left(x + \frac{1}{2}\right)\left(\log\left(x + \frac{1}{2}\right) - \log x\right)$

(62) $\lim_{x \to 0} \left[\left(\frac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}\right)^2 + \frac{1 - \cos x}{\cos x \sin^2 x}\right]$

2. Find the values of a, b, c such that $\lim_{x \to 0} \frac{a\sin x - bx + cx^2 + x^3}{2x^2 \log(1+x) - 2x^3 + x^4}$ be finite. Also determine the limit.

3. Find the values of a, b, so that $\lim_{x \to 0} \frac{x(1 + a\cos x) - b\sin x}{x^3} = 1$.

4. Find the values of a, b for which $\lim_{x \to 0} \frac{\sin 3x + ax + bx^3}{x^3} = 0$.

5. Find the values of a, b for which $\lim_{x \to 0} \frac{a\sinh x + b\sin x}{x^3} = \frac{5}{3}$.

6. Find the values of a, b for which $\lim_{x \to 0} \frac{a\sin^2 x + b\log(\cos x)}{x^4} = \frac{1}{2}$.

7. $\lim_{x \to a} \sin^{-1}\left(\sqrt{\frac{a-x}{a+x}}\right) \operatorname{cosec}\left(\sqrt{a^2 - x^2}\right)$.

8. Find the value of a for which $\lim_{x \to a} \frac{\sin 2x + a\sin x}{x^3}$ is finite. Hence find the limit.

9. Find the values of a, b for which

(1) $\lim_{x \to 0} \frac{\sin x + ax + bx^3}{x^3} = 0$.

(2) $\lim_{x \to 0} \frac{x(1 - a\cos x) + b\sin x}{x^3} = \frac{1}{3}$

(3) $\lim_{x \to 0} \frac{a\sin^2 x + b\cos(\log x)}{x^4} = -\frac{1}{2}$.

CONIC SECTIONS

In this section we give geometric definitions of parabolas, ellipses, and hyperbolas and derive their standard equations. This curves are called **conic section** or **conics.**

CONIC SECTIONS AND QUADRATIC EQUATIONS

We defined a **circle** as the set of points in a plane whose distance from some fixed center point is a constant radius value. If the center is (h, k) and the radius is a, the standard equation for the circle is $(x-h)^2 + (y-k)^2 = a^2$. It is an example of a conic section, which are the curves formed by cutting a double cone with a plane (Fig. 1); hence the name **conic section.**

We now describe parabolas, ellipses and hyperbolas as the graphs of quadratic equations in the coordinate plane.

■ **Parabolas :**

Definition : Parabola, Focus, Directrix

A set that consists of all the points in a plane equidistant from a given fixed point and a given fixed line in the plane is a **parabola.** The fixed point is the **focus** of the parabola. The fixed line is the **directrix.**

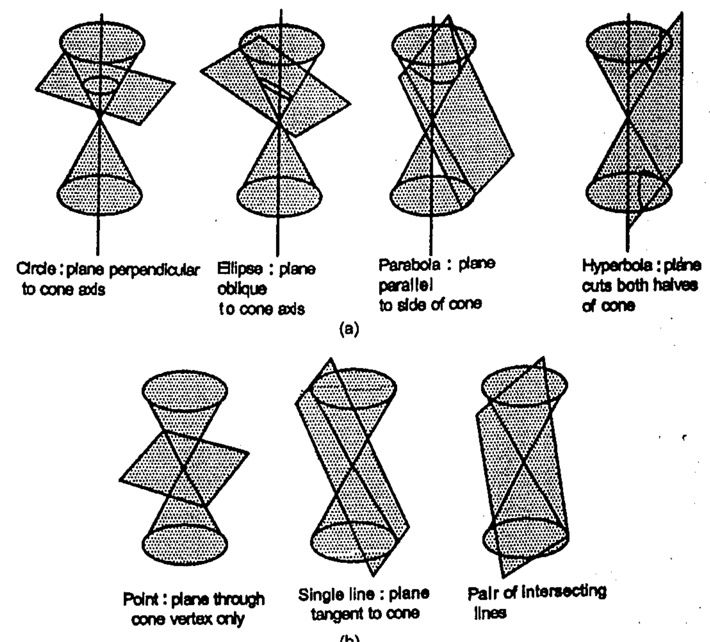


Fig. 1 : The standard conic section (a) are the curves in which a plane cuts a double cone. Hyperbolas come in two parts, called *branches*. The point and lines obtained by passing the plane through the cone's vertex (b) are *degenerate* conic sections.

If the focus F lies on the directrix L, the parabola is the line through F perpendicular to L. We consider this to be a degenerate case and assume henceforth that F does not lie on L.

- A parabola has its simplest equation when its focus and directrix straddle one of the coordinate axes. For example, suppose that the focus lies at the point F(0, p) on the positive Y-axis and that the directrix is the line $y = -p$ (Fig. 2). In the notation of the figure, a point P(x, y) lies on the parabola if and only if PF = PQ. From the distance formula,

$$PF = \sqrt{(x-0)^2 + (y-p)^2}$$

$$= \sqrt{x^2 + (y-p)^2}$$

$$PQ = \sqrt{(x-x)^2 + (y-(-p))^2}$$

$$= \sqrt{(y+p)^2}$$

When equate these expressions, square, and simplify, we get

$$y = \frac{x^2}{4p} \quad \text{or} \quad x^2 = 4py \text{. ... (1)}$$

These equations reveal the parabola's symmetry about the y-axis. We call the y-axis the axis of the parabola (short for "axis of symmetry").

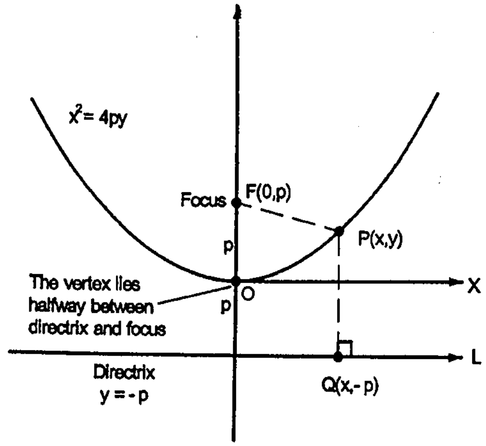


Fig. 2 : The standard form of the parabola $x^2 = 4py$, $p > 0$

The point where a parabola cross its axis is the **vertex**. The vertex of the parabola $x^2 = 4py$ lies at the origin (Fig. 2). The positive number p is the parabola's **focal length.**

If the parabola opens downward, with its focus at (0, $-p$) and its directrix the line $y = p$, then Equation (1) become

$$y = -\frac{x^2}{4p} \quad \text{and} \quad x^2 = -4py$$

Fig. 3. We obtain similar equations for parabolas opening to the right or to the left Fig. 4 and Table-1.

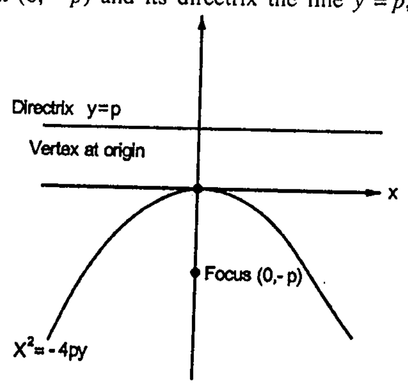


Fig. 3 : The parabola $x^2 = -4py$, $p > 0$

Table-1 : Standard-form equations for parabolas with vertices at the origin ($p > 0$)

Equation	Focus	Directrix	Axis	Opens
$x^2 = 4py$	$(0,\ p)$	$y = -p$	y-axis	Up
$x^2 = -4py$	$(0,\ -p)$	$y = p$	y-axis	Down
$y^2 = 4px$	$(p,\ 0)$	$x = -p$	x-axis	To the right
$y^2 = -4px$	$(-p,\ 0)$	$x = p$	x-axis	To the left

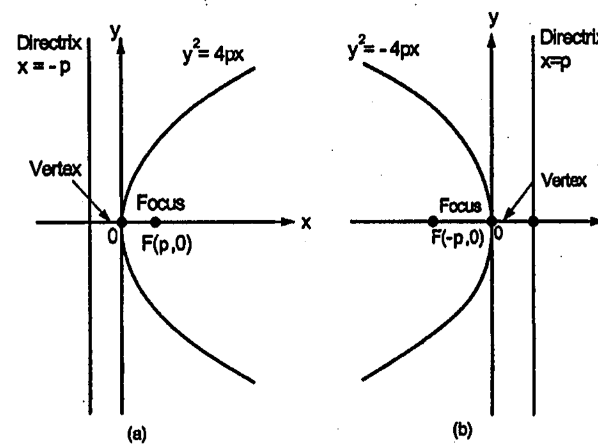


Fig. 4 : (a) The parabola $y^2 = 4px$. (b) The parabola $y^2 = -4px$

Example-1 :

Find the focus and directrix of the parabola $y^2 = 10x$.

Solution :

We find the value of p in the standard equation $y^2 = 4px$.

$$4p = 10, \quad \text{so} \quad p = \frac{10}{4} = \frac{5}{2}$$

Then we find the focus and directrix for this value of p :

$$\text{Focus :} \quad (p,\ 0) = \left(\frac{5}{2}, 0\right)$$

$$\text{Directrix :} \quad x = -p \quad \text{or} \quad x = -\frac{5}{2}$$

■ **Ellipses :**

Definitions : Ellipse, Foci

An **ellipse** is the set of points in a plane whose distances from two fixed points in the plane have a constant sum. The two fixed points are the **foci** of the ellipse.

The quickest way to construct an ellipse uses the definition. Put a loop of string around two tacks F_1 and F_2, pull the string taut with a pencil point P, and move the pencil around to trace a closed curve (Fig. 5). The curve is an ellipse because the sum $PF_1 + PF_2$, being the length of the loop minus the distance between the tacks, remains constant. The ellipse's foci lie at F_1 and F_2.

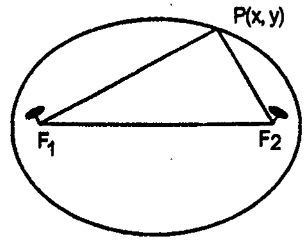


Fig. 5 : One way to draw an ellipse uses two tacks and a loop of string to guide the pencil

Definitions : Focal Axis, Center, Vertices :

The line through the foci of an ellipse is the ellipse's **focal axis.** The point on the axis halfway between the foci is the **center.** The point where the focal axis and ellipse cross are the ellipse's **vertices** (Fig. 6).

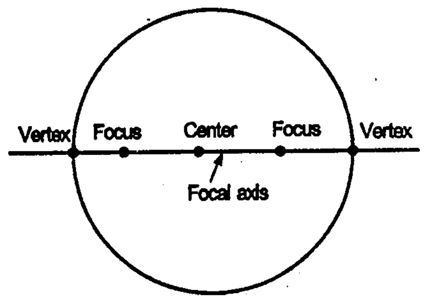


Fig. 6 : Points on the focal axis of an ellipse

If the foci are $F_1(-c, 0)$ and $F_2(c, 0)$ (Fig. 7), and $PF_1 + PF_2$ is denoted by $2a$, then the coordinates of a point P on the ellipse satisfy the equation

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a.$$

$$\Rightarrow \sqrt{(x+c)^2+y^2} = 2a-\sqrt{(x-c)^2+y^2}$$

$$\Rightarrow (x+c)^2+y^2 = [2a-\sqrt{(x-c)^2+y^2}]^2$$

$$\Rightarrow x^2+2xc+c^2+y^2$$

$$= 4a^2-4a\sqrt{(x-c)^2+y^2}+x^2-2xc+c^2+y^2$$

$$\Rightarrow 4xc-4a^2=4a\sqrt{(x-c)^2+y^2}$$

$$\Rightarrow (xc-a^2)^2=a^2[(x-c)^2+y^2]$$

$$\Rightarrow x^2c^2-2xca^2+a^4=a^2x^2-2xca^2+a^2c^2+a^2y^2$$

$$\Rightarrow a^2(a^2-c^2)=x^2(a^2-c^2)+a^2y^2$$

$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1. \qquad \ldots (2)$$

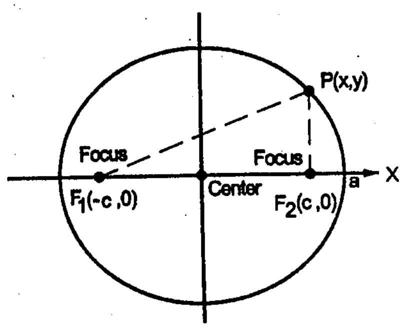


Fig. 7 : The ellipse defined by the equation $PF_1 + PF_2 = 2a$ is the graph of the equation $(x^2/a^2) + (y^2/b^2) = 1$, where $b^2 = a^2 - c^2$.

Since $PF_1 + PF_2$ is greater than the length F_1F_2 (triangle inequality for triangle PF_1F_2) the number $2a$ is greater than $2c$. Accordingly, $a > c$ and the number $a^2 - c^2$ in Equation (2) is positive.

The algebraic steps leading to Equation (2) can be reversed to show that every point P whose coordinates satisfy an equation of this form with $0 < c < a$ also satisfies the equation $PF_1 + PF_2 = 2a$. A point therefore lies on the ellipse if and only if its coordinates satisfy Equation (2).

If $\quad b = \sqrt{a^2-c^2} \qquad \ldots (3)$

then $a^2-c^2=b^2$ and Equation (2) takes the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \ldots (4)$$

Equation (4) reveals that this ellipse is symmetric with respect to the origin and both coordinate axes. It lies inside the rectangle bounded by the lines $x = \pm a$ and $y = \pm b$. It crosses the axes at the point $(\pm a, 0)$ and $(0, \pm b)$. The tangents at these points are perpendicular to the axes because

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y} \quad \text{(obtained from Equation (4) by implicit differentiation)}$$

is zero if $x = 0$ and infinity if $y = 0$.

The **major axis** of the ellipse in Equation (4) is the line segment of length $2a$ joining the points $(\pm a, 0)$. The **minor axis** is the line segment of length joining the points $(0, \pm b)$. The number a itself is the **semimajor axis,** the number b the **semiminor axis.** The number c, found from Equation (3) as $c = \sqrt{a^2-b^2}$, is the **center-to-focus distance** of the ellipse.

Example-2 :

Find semimajor axis, semiminor axis, center-to-focus distance, foci, vertices, and center for

(1) $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$

Solution : From Fig. 8

Semimajor axis : $a = \sqrt{16} = 4$,

Semiminor axis : $b = \sqrt{9} = 3$

Center-to-focus distance : $c = \sqrt{16-9} = \sqrt{7}$

Foci : $(\pm c, 0) = (\pm\sqrt{7}, 0)$

Vertices : $(\pm a, 0)$, $(\pm 4, 0)$

Center : $(0, 0)$.

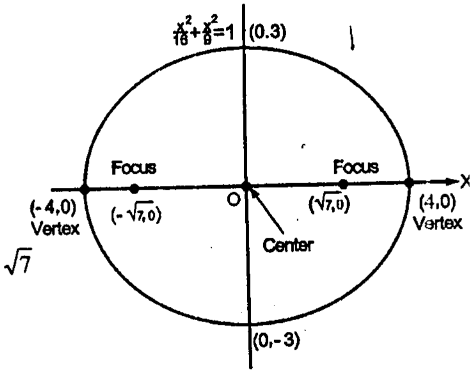


Fig. 8 : An ellipse with its major axis horizontal

(2) $\frac{x^2}{9}+\frac{y^2}{16}=1$

Solution :

The ellipse

$$\frac{x^2}{9}+\frac{y^2}{16}=1$$

obtained by interchanging x and y in Equation (5), has its major axis vertical instead of horizontal (Fig. 9). With a^2 still equal to 16 and b^2 equal to 9, we have

Semimajor axis : $a = \sqrt{16} = 4$,

Semiminor axis : $b = \sqrt{9} = 3$

Center-to-focus distance : $c = \sqrt{16-9} = \sqrt{7}$

Foci : $(0, \pm c) = (0, \pm\sqrt{7})$

Vertices : $(0, \pm a)$, $(0, \pm 4)$

Center : $(0, 0)$.

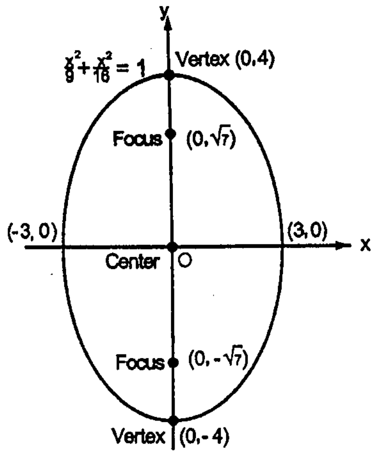


Fig. 9 : An ellipse with its major axis vertical

■ **Standard-Form Equations for Ellipses Centered at the Origin :**

Foci on the x-axis : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a > b)$

Center-to-focus distance : $c = \sqrt{a^2-b^2}$

Foci : $(\pm c, 0)$

Vertices : $(\pm a, 0)$

Foci on the y-axis : $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ $(a > b)$

Center-to-focus distance : $c = \sqrt{a^2-b^2}$

Foci : $(0, \pm c)$

Vertices : $(0, \pm a)$

In each case, a is the semimajor axis and b is the semiminor axis.

■ **Hyperbolas :**

Definitions : Hyperbola, Foci

A **hyperbola** is the set of points in a plane whose distances from two fixed points in the plane have a constant difference. The two fixed points are **foci** of the hyperbola.

If the foci are $F_1(-c, 0)$ and $F_2(c, 0)$ (Fig. 10) and the constant difference is $2a$, then a point (x, y) lies on the hyperbola if and only if

$$\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=\pm 2a. \quad \text{... (7)}$$

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining redical and square again, obtaining.

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1 \quad \text{... (8)}$$

So far, this looks just like the equation for an ellipse. But now a^2-c^2 is negative because $2a$, being the difference of two sides of triangle PF_1F_2, is less than $2c$, the third side.

The algebraic steps leading to Equation (8) can be reversed to show that every point P whose coordinates satisfy an equation of this form with $0 < a < c$ also satisfies Equation (7). A point therefore lies on the hyperbola if and only if its coordinates satisfy Equation (8).

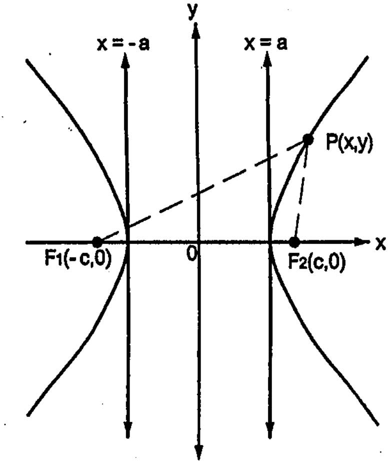


Fig. 10 : Hyperbolas have two branches. For points on the right-hand branch of the hyperbola shown here, $PF_1 - PF_2 = 2a$. For points on the left-hand branch $PF_2 - PF_1 = 2a$. We then let $b = \sqrt{c^2-a^2}$.

If we let b denote the positive square root of c^2-a^2,

$$b = \sqrt{c^2-a^2}, \quad \text{... (9)}$$

then $a^2-c^2=-b^2$ and Equation (8) takes the more compact form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \quad \text{... (10)}$$

The differences between Equation (10) and the equation for an ellipse (Equation 4) are the minus sign and the new relation

$c^2 = a^2+b^2$. From Equation (9)

Like the ellipse, the hyperbola is symmetric with respect to the origin and coordinate axes. It crosses the x-axis at the point $(\pm a, 0)$. The tangents at these points are vertical because

$$\frac{dy}{dx}=\frac{b^2x}{a^2y} \quad \text{(Obtained from Equation (10) by implicit differentiation)}$$

is infinite when $y = 0$. The hyperbola has no y-intercepts; in fact, no part of the curve lies between the lines $x = -a$ and $x = a$.

Definitions : Focal Axis, Center, Vertices

The line through the foci of a hyperbola is the **focal axis**. The point on the axis halfway between the foci is the hyperbola's **center**. The points where the focal axis and hyperbola cross are the **vertices** (Fig. 11).

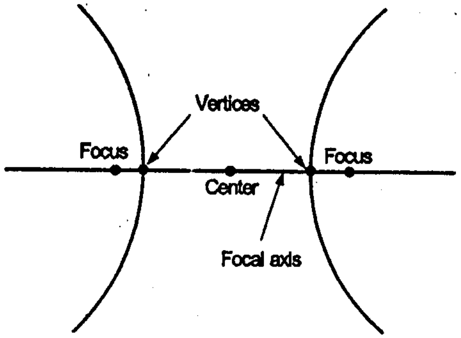


Fig. 11 : Points on the focal axis of a hyperbola

Asymptotes of Hyperbolas and Graphing :

If we solve Equation (10) for y we obtain

$$y^2 = b^2\left(\frac{x^2}{a^2} - 1\right)$$

$$= \frac{b^2}{a^2}x^2\left(1 - \frac{a^2}{x^2}\right)$$

or, taking square roots,

$$y = \pm\frac{b}{a}x\sqrt{1 - \frac{a^2}{x^2}}$$

As $x \to \pm\infty$, the factor $\sqrt{1 - \frac{a^2}{x^2}}$ approaches 1, and the factor $\pm\left(\frac{b}{a}\right)x$ is dominant. Thus the lines

$$y = \pm\frac{b}{a}x$$

are the **asymptotes** of the hyperbola defined by Equation (10). The asymptotes give the guidance we need to graph hyperbolas quickly. The fastest way to find the equations of the asymptotes is to replace the 1 in Equation (10) by 0 and solve the new equation for y:

$$\underbrace{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}_{\text{hyperbola}} \rightarrow \underbrace{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0}_{\text{0 for 1}} \rightarrow \underbrace{y = \pm\frac{b}{a}x}_{\text{asymptotes}}$$

Standard-From Equations for Hyperbolas Centered at the Origin :

Foci on the x-axis : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Center-to-focus distance : $c = \sqrt{a^2 + b^2}$

Foci : $(\pm c,\ 0)$

Vertices : $(\pm a,\ 0)$

Asymptotes : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ or $y = \pm\frac{b}{a}x$

Foci on the y-axis : $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Center-to-focus distance : $c = \sqrt{a^2 + b^2}$

Foci : $(0,\ \pm c)$

Vertices : $(0,\ \pm a)$

Asymptotes : $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 0$ or $y = \pm\frac{a}{b}x$

Notice the difference in the asymptote equations (b/a in the first, a/b in the second).

Example-3 :

Find center-to-focus distance, foci, vertices, center and asymptotoes for

(1) $\frac{x^2}{4} - \frac{y^2}{5} = 1$

Solution :

The equation

$$\frac{x^2}{4} - \frac{y^2}{5} = 1, \qquad \text{... (11)}$$

is Equation (10) with $a^2 = 4$ and $b^2 = 5$ (Fig. 12).

We have

Center-to-focus distance :

$$c = \sqrt{a^2 + b^2} = \sqrt{4 + 5} = 3$$

Foci : $(\pm c,\ 0) = (\pm 3,\ 0)$

Vertices : $(\pm a,\ 0)$, $(\pm 2,\ 0)$

Center : $(0,\ 0)$.

Asymptotes : $\frac{x^2}{4} - \frac{y^2}{5} = 0$ or $y = \pm\frac{\sqrt{5}}{2}x$

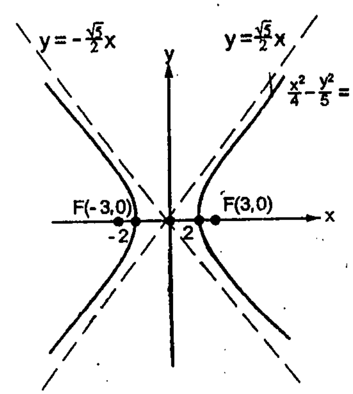


Fig. 12 : The hyperbola and its asymptotes in Example-1

(2) $\frac{y^2}{4} - \frac{x^2}{5} = 1$

Solution :

The hyperbola

$$\frac{y^2}{4} - \frac{x^2}{5} = 1,$$

obtained by interchanging x and y in Equation (11), has its vertices on the y-axis instead of the x-axis (Fig. 13). With a^2 still equal to 4 and b^2 equal to 5, we have

Center-to-focus distance : $c = \sqrt{a^2 + b^2} = \sqrt{4+5} = 3$

Foci : $(0, \pm c) = (0, \pm 3)$

Vertices : $(0, \pm a) = (0, \pm 2)$

Center : $(0, 0)$.

Asymptotes : $\frac{y^2}{4} - \frac{x^2}{5} = 0$ or $y = \pm \frac{2}{\sqrt{5}} x$

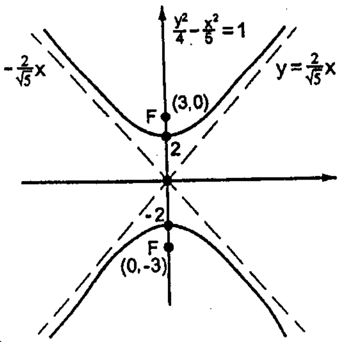


Fig. 13 : The hyperbola and its asymptotes

Reflective Properties :

The chief applications of parabolas involve their use as reflectors of light and radio waves. Rays originating at a parabola's focus are reflected out of the parabola parallel to the parabola's axis (Fig. 14). Moreover, the time any ray takes from the focus to a line parallel to the parabola's directrix (thus perpendicular to its axis) is the same for each of the rays. These properties are used by flashlight, headlight, and spotlight reflectors and by microwave broadcast antennas.

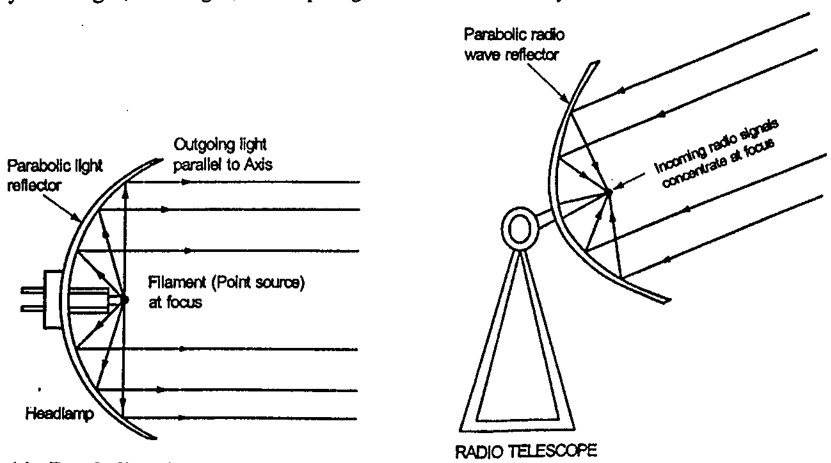


Fig. 14 : Parabolic reflectors can generate a beam of light parallel to the parabola's axis from a source at the focus; or they can receive rays parallel to the axis and concentrate them at the focus.

If an ellipse is revolved about its major axis to generate a surface (the surface is called an *ellipsoid*) and the interior is silvered to produce a mirror, light from one focus will be reflected to the other focus (Fig. 15). Ellipsoids reflect sound the same way, and this property is used to construct *whispering galleries*, rooms in which a person standing at one focus can hear a whisper from the other focus. (Statuary Hall in the U.S. Capitol building is a whispering gallery).

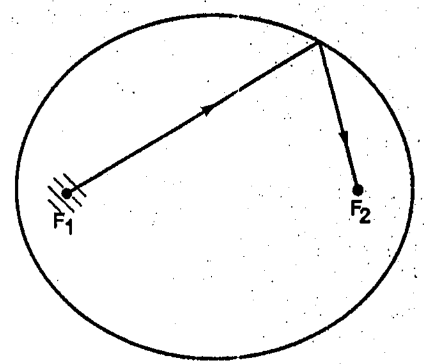


Fig. 15 : An elliptical mirror (shown here in profile) reflects light from one focus to the other

Light directed toward one focus of a hyperbolic mirror is reflected toward the other focus. This property of hyperbolas is combined with the reflective properties of parabolas and ellipses in designing some modern telescopes. In Fig. 16 starlight reflects off a parimar parabolic mirror toward the mirror's focus F_P. It is then reflected by a small hyperbolic mirror, whose focus is $F_H = F_P$, toward the second focus of the hyperbola, $F_E = F_H$. Since this focus is shared by an ellipse, the light is reflected by the elliptical mirror to the ellipse's second focus to be seen by an observer.

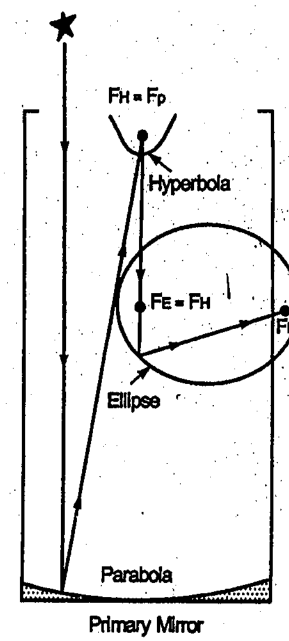


Fig. 16 : Schematic drawing of a reflecting telescope

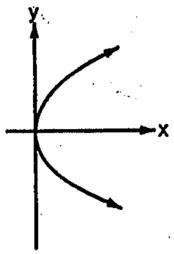
EXERCISE-1

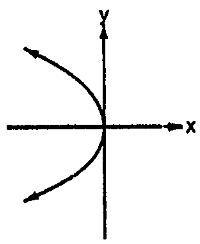
Identifying Graphs :

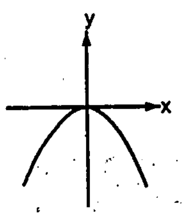
Match the parabolas in Exercise 1–4 with the following equations.

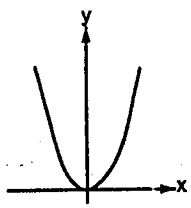
$x^2 = 2y,\ x^2 = -6y,\ y^2 = 8x,\ y^2 = -4x$

Then find the parabola's focus and directrix.

1. 

2. 

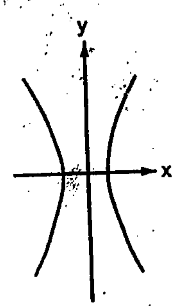
3. 

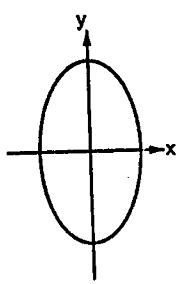
4. 

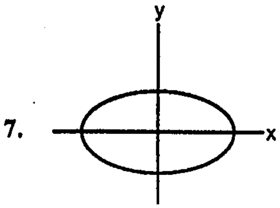
Match each conic section in Exercises 5-8 with one of these equation :

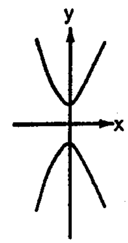
$\frac{x^2}{4} + \frac{y^2}{9} = 1,\ \frac{x^2}{2} + y^2 = 1,$

$\frac{y^2}{4} - x^2 = 1,\ \frac{x^2}{4} - \frac{y^2}{9} = 1.$

5. 

6. 

7. 

8. 

Parabolas :

Exercises 9–16 give equations of parabolas. Find each parabola's focus and directrix. Then sketch the parabola. Include the focus and directrix in your sketch.

9. $y^2 = 12x$
10. $x^2 = 6y$
11. $x^2 = -8y$
12. $y^2 = -2x$
13. $y = 4x^2$
14. $y = -8x^2$
15. $x = -3y^2$
16. $x = 2y^2$

Ellipses :

Exercises 17–24 give equations for ellipses. Put each equation in standard form. Then sketch the ellipse. Include the foci in your sketch.

17. $16x^2 + 25y^2 = 400$
18. $7x^2 + 16y^2 = 112$
19. $2x^2 + y^2 = 2$
20. $2x^2 + y^2 = 4$
21. $3x^2 + 2y^2 = 6$
22. $9x^2 + 10y^2 = 90$
23. $6x^2 + 9y^2 = 54$
24. $169x^2 + 25y^2 = 4225$

Exercises 25 and 26 give information about the foci and vertices of ellipses centered at the origin of the *xy*-plane. In each case, find the ellipse's standard-form equation from the given information.

25. Foci : $(\pm\sqrt{2},\ 0)$; Vertices : $(\pm 2,\ 0)$
26. Foci : $(0,\ \pm 4)$; Vertices : $(0,\ \pm 5)$

Hyperbolas :

Exercises 27–34 give equations for hyperbolas. Put each equation in standard form and find the hyperbola's asymptotes. Then sketch the hyperbola. Include the asymptotes and foci in your sketch.

27. $x^2 - y^2 = 1$
28. $9x^2 - 16y^2 = 144$
29. $y^2 - x^2 = 8$
30. $y^2 - x^2 = 4$
31. $8x^2 - 2y^2 = 16$
32. $y^2 - 3x^2 = 3$
33. $8y^2 - 2x^2 = 16$
34. $64y^2 - 36y^2 = 2304$

Exercise 35–38 give information about the foci, vertices, and asymptotes of hyperbolas centered at the origin of the xy-plane. In each case, find the hyperbola's standard-form equation from the information given.

35. Foci : $(0, \pm\sqrt{2})$; Asymptotes : $y = \pm x$
36. Foci : $(\pm 2, 0)$; Asymptotes : $y = \pm \frac{1}{\sqrt{3}} x$
37. Vertices : $(\pm 3, 0)$; Asymptotes : $y = \pm \frac{4}{3} x$
38. Vertices : $(0, \pm 2)$; Asymptotes : $y = \pm \frac{1}{2} x$

Shifting Conic Sections :

39. The parabola $y^2 = 8x$ is shifted down 2 units and right 1 unit to generate the parabola $(y+2)^2 = 8(x-1)$.
    (a) Find the new parabola's vertex, focus, and directrix.
    (b) Plot the new vertex, focus, and directrix, and sketch in the parabola.
40. The parabola $x^2 = -4y$ is shifted left 1 unit and up 3 units to generate the parabola $(x+1)^2 = -4(y-3)$.
    (a) Find the new parabola's vertex, focus and directrix.
    (b) Plot the new vertex, focus, and directrix, and sketch in the parabola.
41. The ellipse $(x^2/16) + (y^2/9) = 1$ is shifted 4 units to the right and 3 units up to generate the ellipse
    $$\frac{(x-4)^2}{16} + \frac{(y-3)^2}{9} = 1$$
    (a) Find the foci, vertices, and center of the new ellipse.
    (b) Plot the new foci, vertices, and center, and sketch in the new ellipse.
42. The ellipse $(x^2/9) + (y^2/25) = 1$ is shifted 3 units to the left and 2 units down to generate the ellipse
    $$\frac{(x+3)^2}{9} + \frac{(y+2)^2}{25} = 1.$$
    (a) Find the foci, vertices, and center of the new ellipse.
    (b) Plot the new foci, vertices, and center, and sketch in the new ellipse.
43. The hyperbola $(x^2/16) - (y^2/9) = 1$ is shifted 2 units to the right to generate the hyperbola
    $$\frac{(x-2)^2}{16} + \frac{y^2}{9} = 1.$$
    (a) Find the foci, vertices, and asymptotes of the new ellipse.
    (b) Plot the new foci, vertices, and asymptotes, and sketch in the hyperbola.

44. The hyperbola $(y^2/4) - (x^2/5) = 1$ is shifted 2 units down to generate the hyperbola

$$\frac{(y+2)^2}{4} - \frac{x^2}{5} = 1.$$

(a) Find the center, foci, vertices, and asymptotes of the new hyperbola.

(b) Plot the new center, foci, vertices, and asymptotes, and sketch in the hyperbola.

Exercise 45–48 give equations for parabolas and tell how many units up or down and to the right or left each parabola is to be shifted. Find an equation for the new parabola, and find the new vertex, focus, and directrix.

45. $y^2 = 4x$, left 2, down 3

46. $y^2 = -12x$, right 4, up 3

47. $x^2 = 8y$, right 1, down 7

48. $x^2 = 6y$, left 3, down 2

Exercise 49–52 give equation for ellipses and tell how many units up or down and to the right or left each ellipse is to be shifted. Find an equation for the new ellipse, and find the new foci, vertices, and center.

49. $\frac{x^2}{6} + \frac{y^2}{9} = 1$, left 2, down 1

50. $\frac{x^2}{2} + y^2 = 1$, right 3, up 4

51. $\frac{x^2}{3} + \frac{y^2}{2} = 1$, right 2, up 3

52. $\frac{x^2}{16} + \frac{y^2}{25} = 1$, left 4, down 5

Exercise 53–56 give equations for hyperbolas and tell how many units up or down and to the right or left each hyperbola is to be shifted. Find an equation for the new hyperbola, and find the new center, foci, vertices, and asymptotes.

53. $\frac{x^2}{4} - \frac{y^2}{5} = 1$, right 2, up 2

54. $\frac{x^2}{16} - \frac{y^2}{9} = 1$, left 2, down 1

55. $y^2 - x^2 = 1$, left 1, down 1

56. $\frac{y^2}{3} - x^2 = 1$, right 1, up 3

Find the center, foci, vertices, asymptotes, and radius, as appropriate, of the conic sections in Exercises 57–68.

57. $x^2 + 4x + y^2 = 12$

58. $2x^2 + 2y^2 - 28x + 12y + 114 = 0$

59. $x^2 + 2x + 4y - 3 = 0$

60. $y^2 - 4y - 8x - 12 = 0$

61. $x^2 + 5y^2 + 4x = 1$

62. $9x^2 + 6y^2 + 36y = 0$

63. $x^2 + 2y^2 - 2x - 4y = -1$

64. $4x^2 + y^2 + 8x - 2y = -1$

65. $x^2 - y^2 - 2x + 4y = 4$

66. $x^2 - y^2 + 4x - 6y = 6$

67. $2x^2 - y^2 + 6y = 3$

68. $y^2 - 4x^2 + 16x = 24$

Inequalities :

Sketch the regions in the *xy*-plane whose coordinates satisfy the inequalities or pairs of inequalities in Exercise 69–74.

69. $9x^2 + 16y^2 \le 144$

70. $x^2 + y^2 \ge 1$ and $4x^2 + y^2 \le 4$

71. $x^2 + 4y^2 \ge 4$ and $4x^2 + 9y^2 \le 36$

72. $(x^2 + y^2 - 4)(x^2 + 9y^2 - 9) \le 0$

73. $4y^2 - x^2 \ge 4$

74. $|x^2 - y^2| \le 1$

CLASSIFYING CONIC SECTIONS BY ECCENTRICITY

We now show how to associate with each conic section a number called the conic section's ***eccentricity***. The eccentricity reveals the conic section's type (circle, ellipse, parabola, or hyperbola) and, in the case of ellipses and hyperbolas, describes the conic section's general proportions.

Eccentricity :

Although the center-to-focus distance c does not appear in the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ (a > b)$$

for an ellipse, we can still determine c from the equation $c = \sqrt{a^2 - b^2}$. If we fix a and vary c over the interval $0 \le c \le a$, the resulting ellipses will vary shape (Fig. 17). They are circles if $c = 0$ (so that $a = b$) and flatten as c increases. If $c = a$, the foci and vertices overlap and the ellipse degenerates into a line segment.

We use the ratio of c to a to describe the various shapes the ellipse can take. We call this ratio the ellipse's eccentricity.

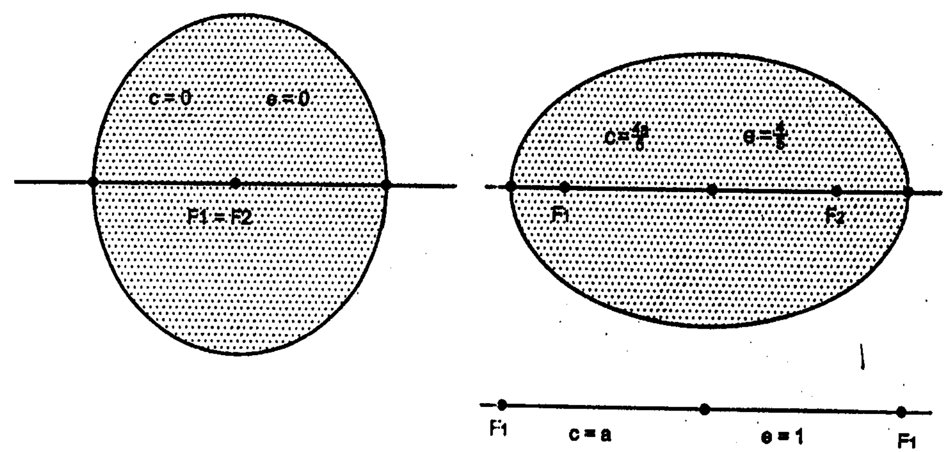


Fig. 17 : The ellipse changes from a circle to a line segment as *c* increases from 0 to *a*.

Definition : Eccentricity of an Ellipse

The eccentricity of the ellipse $\left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) = 1 \ (a > b)$ is $e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}$.

The planets in the solar system revolve around the sun in (approximate) elliptical orbits with the sun at one focus. Most of the orbits are nearly circular, as can be seen from the eccentricities in Table-2. Pluto has a fairly eccentric orbit, with $e = 0.25$, as does Mercury, with $e = 0.21$. Other members of the solar system have orbits that are even more eccentric.

Icarus, an asteroid about 1 mile wide that revolves around the sun every 409 Earth days, has an orbital eccentricity of 0.83 (Fig. 18).

Table-2 : Eccentricities of planetary orbits

Mercury	0.21	Saturn	0.06
Venus	0.01	Uranus	0.05
Earth	0.02	Neptune	0.01
Mars	0.09	Pluto	0.25
Jupiter	0.05		

The planets in the solar system revolve around the sun in (approximate) elliptical orbits with the sun at one focus. Most of the orbits are nearly circular, as can be seen from the eccentricities in Table-2. Pluto has a fairly eccentric orbit, with $e = 0.25$, as does Mercury, with $e = 0.21$. Other members of the solar system have orbits that are even more eccentric. Icarus, as asteroid about 1 mile wide that revolves around the sun every 409 Earth days, has an orbital eccentricity of 0.83 (Fig. 18).

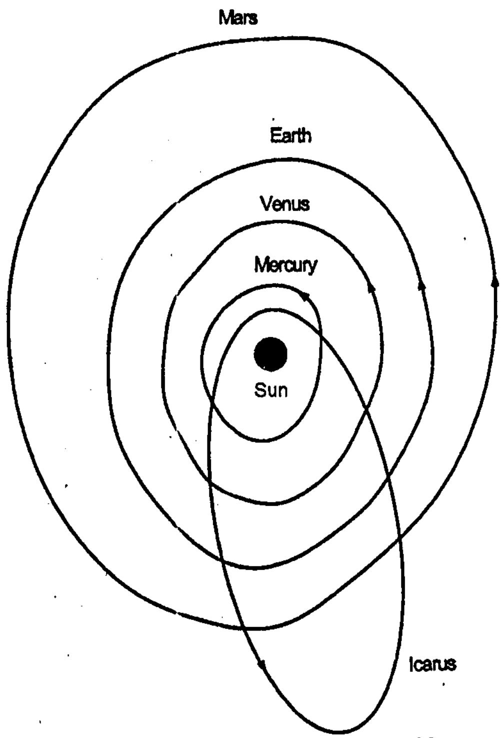


Fig. 18 : The orbit of the asteroid Icarus is highly eccentric. Earth's orbit is so nearly circular that its foci lie inside the sun

Example-1 : Halley's Comet

The orbit of Halley's comet is an ellipse 36.18 astronomical units long by 9.12 astronomical units wide. (One *astronomical unit* [AU] is 149,597,870 km, the semimajor axis of Earth's orbit.) Its eccentricity is

$$e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{\left(\frac{36.18}{2}\right)^2 - \left(\frac{9.12}{2}\right)^2}}{\left(\frac{1}{2}\right)(36.18)} = \frac{\sqrt{(18.09)^2 - (4.56)^2}}{18.09} \approx 0.97.$$

Whereas a parabola has one focus and one directrix, each **ellipse** has two foci and two **directrices**. These are the lines perpendicular to the major axis at distance $\pm a/e$ from the center. The parabola has the property that

$$PF = 1 \cdot PD \quad \ldots (1)$$

for any point P on it, where F is the focus and D is the point nearest P on the directrix. For an ellipse, it can be shown that the equations that replace Equation (1) are

$$PF_1 = e \cdot PD_1, \quad PF_2 = e \cdot PD_2 \quad \ldots (2)$$

Here, e is the eccentricity, P is any point on the ellipse, F_1 and F_2 are the foci, and D_1 and D_2 are the points on the directrixes nearest P (Fig. 19).

In both Equations (2) the directrix and focus must correspond; that is, if we use the distance from P to F_1, we must also use the distance from P to the directrix at the same end of the ellipse. The directrix $x = -a/e$ corresponds to $F_1(-c, 0)$, and the directrix $x = a/e$ corresponds to $F_2(c, 0)$.

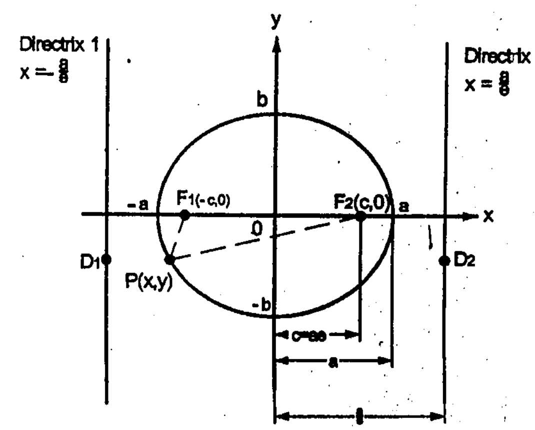


Fig. 19 : The foci and directrices of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$. Directrix 1 corresponds to focus F_1, and directrix 2 to focus F_2

The eccentricity of a hyperbola is also $e = c/a$, only in this case c equals $\sqrt{a^2 + b^2}$ instead of $\sqrt{a^2 - b^2}$. In contrast to the eccentricity of an ellipse, the eccentricity of a hyperbola is always greater than 1.

Definition : Eccentricity of Hyperbola

The eccentricity of the hyperbola $\left(\frac{x^2}{a^2}\right) - \left(\frac{y^2}{b^2}\right) = 1$ is $e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}$.

In both ellipse and hyperbola, the eccentricity is the ratio of the distance between the foci to the distance between the vertices (because $c/a = 2c/2a$).

$$\text{Eccentricity} = \frac{\text{distance between foci}}{\text{distance between vertices}}$$

In an ellipse, the foci are closer together than the vertices and the ratio is less than 1. In a hyperbola, the foci are farther apart than the vertices and the ratio is greater than 1.

Example-2 :

Locate the vertices of an ellipse of accentricity 0.8 whose foci lie at the points $(0, \pm 7)$.

Solution :

Since $e = c/a$, the vertices are the points $(0, \pm a)$ where

$$a = \frac{c}{e} = \frac{7}{0.8} = 8.75$$

or $(0, \pm 8.75)$

Example-3 :

Find the eccentricity of the hyperbola $12x^2 - 27y^2 = 108$.

Solution :

We divide both sides of the hyperbola's equation by 108 to put in standard form

$$\frac{12x^2}{108} - \frac{27y^2}{108} = 1 \quad \text{and} \quad \frac{x^2}{9} - \frac{y^2}{4} = 1.$$

With $a^2 = 9$ and $b^2 = 4$, we find that $c = \sqrt{a^2 + b^2} = \sqrt{9+4} = \sqrt{13}$, so

$$e = \frac{c}{a} = \frac{\sqrt{13}}{3}.$$

As with the ellipse, it can be shown that the lines $x = \pm a/e$ act as **directrices** for the **hyperbola** and that

$$PF_1 = e \cdot PD_1, \quad PF_2 = e \cdot PD_2 \qquad \text{... (3)}$$

Here, P is any point on the hyperbola, F_1 and F_2 are the foci, and D_1 and D_2 are the points nearest P on the directrices (Fig. 20).

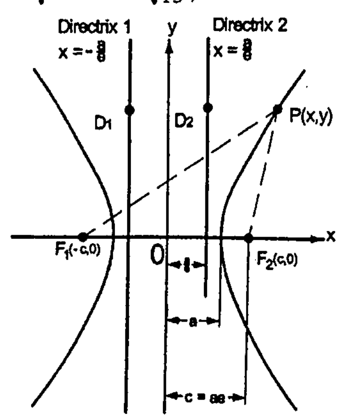


Fig. 20 : The foci and directrices of the ellipse $(x^2/a^2) - (y^2/b^2) = 1$. No matter where P lies on the hyperbola, $PF_1 = e \cdot PD_1$ and $PF_2 = e \cdot PD_2$

To complete the picture, we define the eccentricity of a parabola to be $e = 1$. Equations(1) to (3) then have the common form $PF = e \cdot PD$.

Definition : Eccentricity of a Parabola

The **eccentricity** of a parabola is $e = 1$.

The "focus-directrix" equation $PF = e \cdot PF$ unites the parabola, ellipse, and hyperbola in the following way. Suppose that the distance PF of a point P from a fixed point F (the focus) is a constant multiple of its distance from a fixed line (the directrix). That is, suppose

$$PF = e \cdot PD; \qquad \text{... (4)}$$

where e is the constant of proportionality. Then the path traced by P is

(a) a **parabola** if $e = 1$,

(b) an **ellipse** of eccentricity e if $e < 1$, and

(c) a **hyperbola** of eccentricity e if $e > 1$.

There are no coordinates in Equation (4) and when we try to translate it into coordinate form it translates in different ways, depending on the size of e. At least, that is what happens in Cartesian coordinates. However, in polar coordinates, as we will see in next unit, the equation $PF = e \cdot PD$ translates into a single equation regardless of the value of e, an equation so simple that it has been the equation of choice of astronomers and space scientists for nearly 300 years.

Given the focus and corresponding directrix of a hyperbola centered at the origin and with foci on the x-axis, we can use the dimensions shown in Fig. 20 to find e. Knowing e, we can derive a Cartesian equation for the hyperbola from the equation $PF = e \cdot PD$, as in the next example. We can find equations for ellipses centered at the origin and with foci on the x-axis in a similar wa, using the dimensions shown in Fig. 19.

Example-4 :

Find a Cartesian equation for the hyperbola centered at the origin that has a focus (3, 0) and the line $x = 1$ as the corresponding directrix.

Solution :

We first use the dimensions shown in Fig. 20 to find the hyperbola's eccentricity. The focus is

$$(c, 0) = (3, 0) \quad \text{so} \quad c = 3,$$

The directrix is the line

$$x = \frac{a}{e} = 1, \quad \text{so} \quad a = e.$$

When combined with the equation $e = c/a$ that defines eccentricity, these results give

$$e = \frac{c}{a} = \frac{3}{e}, \quad \text{so} \quad e^2 = 3 \text{ and } e = \sqrt{3}$$

Knowing e, we can now derive the equation we want from the equation $PF = e \cdot PD$. In the notation of Fig. 21, we have

$$PF = e \cdot PD \qquad \text{(equation (4))}$$

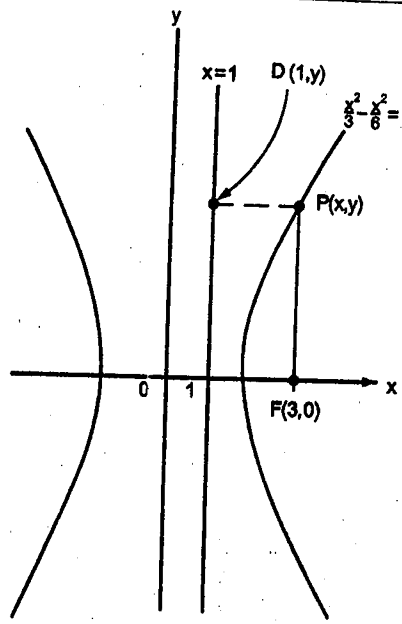


Fig. 21 : The hyperbola and directrix in Example-4

$$\sqrt{(x-3)^2+(y-0)^2} = \sqrt{3}\,|\,x-1\,| \qquad (e=\sqrt{3})$$

$$x^2-6x+9+y^2 = 3(x^2-2x+1)$$

$$2x^2-y^2 = 6$$

$$\frac{x^2}{3}-\frac{y^2}{6} = 1.$$

EXERCISE-2

Ellipses :

In Exercise 1–8, find the eccentricity of the ellipse. Then find and graph the ellipse's foci and directrices.

1. $16x^2+25y^2=400$
2. $7x^2+16y^2=112$
3. $2x^2+y^2=2$
4. $2x^2+y^2=4$
5. $3x^2+2y^2=6$
6. $9x^2+10y^2=90$
7. $6x^2+9y^2=54$
8. $169x^2+25y^2=4225$

Exercise 9–12 give the foci or vertices and the eccentricities of ellipses centered at the origin of the xy-plane. In each case, find the ellipses' standard-form equation.

9. Foci : $(0, \pm 3)$, Eccentricity : 0.5
10. Foci : $(\pm 8, 0)$, Eccentricity : 0.2
11. Vertices : $(0, \pm 70)$, Eccentricity : 0.1
12. Vertices : $(\pm 10, 0)$, Eccentricity : 0.24

Exercise 13–16 give foci and corresponding directrices of ellipses centered at the origin of the xy-plane. In each case, use the dimensions in Fig. 19 to find the eccentricity of the ellipse. Then find the ellipe's standard form equation.

13. Focus : $(\sqrt{5}, 0)$, Directrix : $x=\frac{9}{\sqrt{5}}$
14. Focus : (4, 0), Directrix : $x=\frac{16}{3}$
15. Focus : (– 4, 0), Directrix : $x = -16$
16. Focus : $(-\sqrt{2}, 0)$, Directrix : $x=-2\sqrt{2}$
17. Draw an ellipse of eccentricity 4/5. Explain your procedure.
18. Draw the orbit of Pluto (eccentricity 0.25) to scale. Explain your procedure.
19. The endpoints of the major and minor axes of an ellipse are (1, 1), (3, 4), (1, 7) and (– 1, 4). Sketch the ellipse, give its equation in standard form, and find its foci, eccentricity, and directrices.
20. Find an equation for the ellipse of eccentricity 2/3 that has the line $x = 9$ as a directrix and the point (4, 0) as the corresponding focus,.
21. What values of the constants a, b and c make the ellipse

$$4x^2+y^2+ax+by+c=0$$

lie tangent to the x-axis at the origin and pass through the point (– 1, 2) ? What is the eccentricity of the ellipse ?

Hyperbolas :

In Exercises 22–29, find the eccentricity of the hyperbola. Then find and graph the hyperbola's foci and directrices.

22. $x^2-y^2=1$
23. $9x^2-16y^2=144$
24. $y^2-x^2=8$
25. $y^2-x^2=4$
26. $8x^2-2y^2=16$
27. $y^2-3x^2=3$
28. $8y^2-2x^2=16$
29. $64x^2-36y^2=2304$

Exercises 30–33 give the eccentricities and the vertices or foci of hyperbolas centered at the origin of the xy-plane. In each case, find the hyperbola's standard form equation.

30. Eccentricity : 3, Vertices : $(0, \pm 1)$
31. Eccentricity : 2, Vertices : $(\pm 2, 0)$
32. Eccentricity : 3, Foci : $(\pm 3, 0)$
33. Eccentricity : 5/4, Foci : $(\pm 5, 0)$

Exercises 34–37 give foci and corresponding directrices of hyperbola centered at the origin of the xy-plane. In each case, find the hyperbola's eccentricity. Then find the hyperbola's standard form equation.

34. Focus : (4, 0), Directrix : $x = 2$
35. Focus : $(\sqrt{10}, 0)$, Directrix : $x = \sqrt{2}$
36. Focus : (– 2, 0), Focus : $x=-\frac{1}{2}$
37. Focus : (– 6, 0), Directrix : $x = -2$
38. A hyperbola of eccentricity 3/2 has one focus at (1, – 3). The corresponding directrix is the line $y = 2$. Find an equation for the hyperbola.

QUADRATIC EQUATIONS AND ROTATIONS

In this section, we examine the Cartesian graph of any equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad \text{... (1)}$$

in which A, B and C are not all zero, and show that it is nearly always a conic section. The exceptions are the cases in which there is not graph at all or the graph consists of two parallel lines. It is conventional to call all graphs of Equation (1), curved or not, **quadratic curves.**

The Cross Product Term :

You may have noticed that the term Bxy did not appear in the equations for the conic sections in earlier section. This happened because the axes of the conic sections ran parallel to (in fact, coincided with) the coordinate axes.

To see what happens when the parallelism is absent, let us write an equation for a hyperbola with $a = 3$ and foci at $F_1(-3, -3)$ and $F_2(3, 3)$ (Fig. 22).

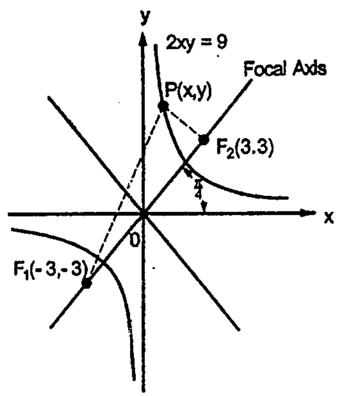


Fig. 22 : The focal axis of the hyperbola 2xy = 9 makes and angle of π/4 radians with the positive x-axis

The equation

$|PF_1 - PF_2| = 2a$ because $|PF_1 - PF_2| = 6$ and

$$\sqrt{(x+3)^2 + (y+3)^2} - \sqrt{(x-3)^2 + (y-3)^2} \pm 6 .$$

When we transpose one radical, square, solve for the radical that still appears, and square again, the equation reduces to

$$2xy = 9, \quad \text{... (2)}$$

a case of Equation (1) in which the cross product term is present. The asymptotes of the hyperbola in Equation (2) are the x and y-axes, and the focal axis makes an angle of π/4 radians with the positive x-axis. As in this example, the cross product term is present in Equation (1) only when the axes of the conic are tilted.

To eliminate the xy-term from the equation of a conic, we rotate the coordinate axes to eliminate the "tilt" in the aces of the conic. The equations for the rotations 'e use are derived in the following way. In the notation of Fig. 23, which shows a counter-clockwise rotation about the origin through an angle α,

$$x = OM = OP\cos(\theta + \alpha) = OP\cos\theta\cos\alpha - OP\sin\theta\sin\alpha$$
$$y = MP = OP\sin(\theta + \alpha) = OP\cos\theta\sin\alpha + OP\sin\theta\sin\alpha \quad \text{... (3)}$$

Since

$$OP\cos\theta = OM' = x'$$

and

$$OP\sin\theta = M'P = y',$$

Equations (3) reduce to the following.

Equations for Rotating Coordinate Axes :

$$x = x'\cos\alpha - y'\sin\alpha \quad \text{... (4)}$$
$$y = x'\cos\alpha - y'\cos\alpha$$

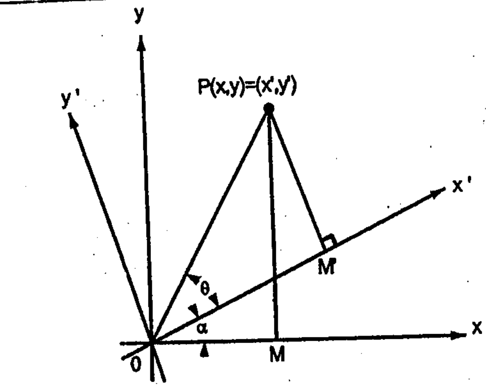


Fig. 23 : A counterclockwise rotation through angle α about the origin

Example-1 :

The x and y-axes rotated through an angle of π/4 radians about the origin. Find an equation for the hyperbola $2xy = 9$ in the new coordinates.

Solution :

Since $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, we substitute

$$x = \frac{x' - y'}{\sqrt{2}}, \quad y = \frac{x' + y'}{\sqrt{2}}$$

from equation (4) into the equation $2xy = 9$ and obtain

$$2\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) = 9$$

$$\Rightarrow \quad x'^2 - y'^2 = 9$$

$$\Rightarrow \quad \frac{x'^2}{9} - \frac{y'^2}{9} = 1$$

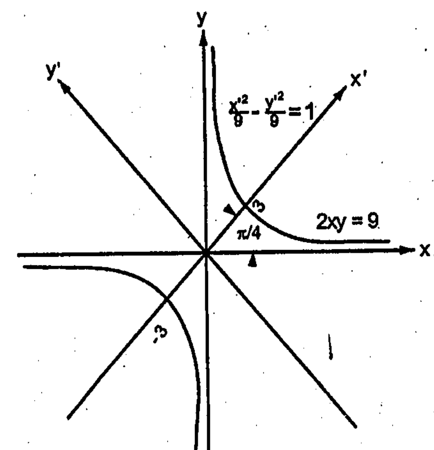


Fig. 24 : The hyperbola in Example-1 (x' and y' are the coordinates)

See Fig. 24.

If we apply Equation (4) to the quadratic equation (1), we obtain a new quadratic equation

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0 \quad \text{... (5)}$$

The new and old coefficients are related by the equations

$$A' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha$$
$$B' = B\cos^2\alpha + (C - A)\sin 2\alpha$$

$C' = A\sin^2\alpha - B\sin\alpha\cos\alpha + C\cos^2\alpha$

$D' = D\cos\alpha + E\sin\alpha$

$E' = -D\sin\alpha + E\cos\alpha$

$F' = F.$

These equations show, among other things, that if we start with an equation for a curve in which the cross product term is present ($B \neq 0$), we can find a rotation angle α that produces an equation in which no cross product term appears ($B' = 0$). To find α, we set $B' = 0$ in the second equation in (6) and solve the resulting equation,

$$B\cos 2\alpha + (C - A)\sin 2\alpha = 0,$$

for α. In practice, this means determining α from one of the two equations.

Angle of Rotation :

$$\cot 2\alpha = \frac{A-C}{B} \quad \text{or} \quad \tan 2\alpha = \frac{B}{A-C} \qquad \text{... (7)}$$

Example-2 :

The coordinate axes are to be rotated through an angle α to produce an equation for the curve

$$2x^2 + \sqrt{3}xy + y^2 - 10 = 0$$

that has no cross product term. Find α and then new equation. Identify the curve.

Solution :

The equation $2x^2 + \sqrt{3}xy + y^2 - 10 = 0$ has $A = 2$, $B = \sqrt{3}$, and $C = 1$. We substitute these values into Equation (7) to find α :

$$\cos 2\alpha = \frac{A-C}{B} = \frac{2-1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

From the right triangle in Fig. 25, we see that one appropriate choice of angle is $2\alpha = \frac{\pi}{3}$, so we take $\alpha = \frac{\pi}{6}$.

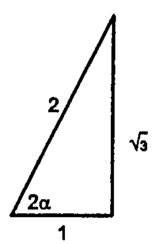


Fig. 25 : This triangle identifies $2\alpha = \cot^{-1}(1/\sqrt{3})$ as $\pi/3$ (Example-2)

Substituting $\alpha = \frac{\pi}{6}$, $A = 2$, $B = \sqrt{3}$, $C = 1$, $D = E = 0$ and $F = -10$ into Equations (6) gives

$$A' = \frac{5}{2}, \quad B' = 0, \quad C' = \frac{1}{2}, \quad D' = E' = 0, \quad F' = -10.$$

Equation (5) the gives

$$\frac{5}{2}x'^2 + \frac{1}{2}y'^2 - 10 = 0 \quad \text{or} \quad \frac{x'^2}{4} + \frac{y'^2}{20} = 1.$$

The curve is an ellipse with foci on the new y'-axis (Fig. 26).

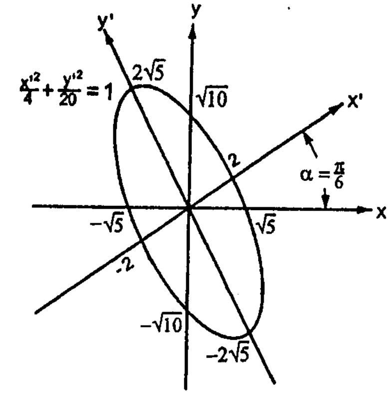


Fig. 26 : The conic section in Example-2

Possible Graphs of Quadratic Equations :

We now return to the graph of the general quadratic equation.

Since aces can always be rotated to eliminate the cross product term, there is no loss of generality in assuming that this has been done and that our equation has the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \qquad \text{... (8)}$$

Equation (8) represents :

(a) a *circle* if $A = C \neq 0$ (special cases : the graph is a point or there is no graph at all);

(b) a *parabola* if Equation (8) is quadratic in one variable and linear in the other;

(c) an *ellipse* if A and C are both positive or both negative (special cases : circles, a single point, or no graph at all);

(d) a *hyperbola* if A and C have opposite signs (special case : a pair of intersecting lines);

(e) a *straight line* if A and C are zero and at least one of D and E is different from zero;

(f) *one or two straight lines* if the left-hand side of Equation (8) can be factored into the product of two linear factors.

See Table-1.3 for examples.

Table-1.3 : Examples of quadratic curves $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

	A	B	C	D	E	F	Equation	Remarks
Circle	1		1			−4	$x^2 + y^2 = 4$	A = C; F < 0
Parabola			1	−9			$y^2 = 9x$	Quadratic in y, linear in x

Ellipse	4		9			–36	$4x^2+9y^2=36$	A, C have same sign, A ≠ C; F < 0
Hyperbola	1		–1			–1	$x^2-y^2=1$	A, C have opposite signs
One line (still a conic section)	1						$x^2=0$	y-axis
Intersecting lines (still a conic section)		1		1	–1	–1	$xy+x-y-1=0$	Factors to $(x-1)(y+1)=0$, so $x=1$, $y=-1$
Parallel lines (not a conic section)	1			–3		2	$x^2-3x+2=0$	Factors to $(x-1)(x-2)=0$, so $x=1$, $x=2$
Point	1		1				$x^2+y^2=0$	The origin
No graph	1					1	$x^2=-1$	No graph

The Discriminant Test

We do not need to eliminate the xy-term from the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

to tell what kind of conic section the equation represents. If this is the only information we want, we can apply the following est instead.

A we have seen, if $B \neq 0$, then rotating the coordinate axes through an angle α that satisfies the equation

$$\cos 2\alpha = \frac{A-C}{B} \quad \text{... (10)}$$

will change Equation (9) into an equivalent form

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0 \quad \text{... (11)}$$

without a cross product term.

Now, the graph of Equation (11) is a (real degenerate)

(a) *parabola* if A′ or C′ = 0; that is, if A′C′ = 0;

(b) *ellipse* if A′ and C′ have the same sign; that is, if A′C′ > 0;

(c) *hyperbola* if A′ and C′ have opposite signs; that is, if A′C′ < 0.

It can also be verified from Equations (6) that for any rotation of axes,

$$B^2 - 4AC = B'^2 - 4A'C' \quad \text{... (12)}$$

This means that the quantity $B^2 - 4AC$ is not changed by rotation. But when we rotate through the angle α given by Equation (10), B′ becomes zero, so

$$B^2 - 4AC = -4A'C'.$$

Since the curve is a parabola if A′C′ = 0, an ellipse if A′C′ > 0, and a hyperbola if A′C′ < 0, the curve must be parabola if $B^2 - 4AC = 0$, and ellipse if $B^2 - 4AC < 0$, and a hyperbola if $B^2 - 4AC > 0$. The number $B^2 - 4AC$ is called the **discriminant** of Equation (9).

The Discriminant Test :

With the understanding that occasional degenerate cases may arise, the quadratic curve $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is

(a) a **parabola** if $B^2 - 4AC = 0$,

(b) an **ellipse** if $B^2 - 4AC < 0$,

(c) a **hyperbola** if $B^2 - 4AC > 0$.

Example-3 :

Use Discriminant test to decide whether following equations represent parabolas, ellipses or hyperbolas.

(a) $3x^2 - 6xy + 3y^2 + 2x - 7 = 0$ represents a parabola because

$$B^2 - 4AC = (-6)^2 - 4 \cdot 3 \cdot 3 = 36 - 36 = 0.$$

(b) $36x^2 + 4y^2 - 72x + 32y - 44 = 0$ represents an ellipse because

$$B^2 - 4AC = 0 - (4)(36)(4) < 0.$$

(c) $xy - y^2 - 5y + 1 = 0$ represents a hyperbola because

$$B^2 - 4AC = (1)^2 - 4(0)(-1) = 1 > 0.$$

EXERCISE-3

Using the Discriminant :

Use the discriminant $B^2 - 4AC$ to decide weather the equations in Exercise 1–16 represent parabolas, ellipses, or hyperbolas.

1. $2x^2 - 8xy + 8y^2 + 2x + 2y = 0$
2. $3x^2 - 18xy + 27y^2 - 5x + 7y = -4$
3. $3x^2 - 7xy + \sqrt{17}y^2 = 1$
4. $2x^2 - \sqrt{15}xy + 2y^2 + x + y = 0$
5. $x^2 - 2xy + y^2 + 2x - y + 2 = 0$
6. $2x^2 - y^2 + 4xy - 2x + 3y = 6$
7. $x^2 + 4xy + 4y^2 - 3x = 6$
8. $x^2 + y^2 + 3x - 2y = 10$
9. $xy + y^2 - 3x = 5$
10. $3x^2 + 6xy + 3y^2 - 4x + 5y = 12$
11. $3x^2 - 5xy + 2y^2 - 7x - 14y = -1$
12. $2x^2 - 36xy + 3y^2 - 4x = 7$
13. $x^2 - 3xy + 3y^2 + 6y = 7$
14. $25x^2 + 21xy + 4y^2 - 350x = 0$
15. $6x^2 + 3xy + 2y^2 + 17y + 2 = 0$
16. $3x^2 + 12xy + 12y^2 + 435x - 9y + 72 = 0$

Rotating Coordinate Axes :

In Exercise 17–26, rotate the coordinate axes to change the given equation into an equation that has no cross product (xy) term. Then identify the graph of the equation. (The new equations will vary with the size and direction of the rotation you use.)

17. $xy = 2$
18. $x^2 + xy + y^2 = 1$
19. $3x^2 + 2\sqrt{3}xy + y^2 - 8\sqrt{3}y = 0$
20. $x^2 - \sqrt{3}xy + 2y^2 = 1$
21. $x^2 - 2xy + y^2 = 2$
22. $3x^2 - 2\sqrt{3}xy + y^2 = 1$
23. $\sqrt{2}x^2 + 2\sqrt{2}xy + \sqrt{2}y^2 - 8x + 8y = 0$
24. $xy - y - x + 1 = 0$
25. $3x^2 + 2xy + 3y^2 = 19$
26. $3x^2 + 4\sqrt{3}xy - y^2 = 7$
27. Find the sine and cosine of an angle in Quadrant I through which the coordinate axes can be rotated to eliminate the cross product term from the equation

    $14x^2 + 16xy + 2y^2 - 10x + 26{,}370y - 17 = 0$

    Do not carry out the rotation.
28. Find the sine and cosine of an angle in Quadrant II through which the coordinate axes can be rotated to eliminate the cross product term from the equation

    $4x^2 - 4xy + y^2 - 8\sqrt{5}x - 16\sqrt{5}y = 0$

    Do not carry out the rotation.

The conic section in Exercises 17–26 were chosen to have rotation angles that were "nice" in the sense that once we knew $\cot 2\alpha$ or $\tan 2\alpha$ we could identify 2α and find $\sin\alpha$ and $\cos\alpha$ from familiar trianlges.

In Exercises 29–34, use a calculator to find an angle α through which the coordinate axes can be rotated to change the given equation into a quadratic equation that has no cross product term. Then find $\sin\alpha$ and $\cos\alpha$ to two decimal places and use Equation (6) top find the coefficients of the new equation to the nearest decimal place. In each case, say whether the conic section is an ellipse, a hyperbola, or a parabola.

29. $x^2 - xy + 3y^2 + x - y - 3 = 0$
30. $2x^2 + xy - 3y^2 + 3x - 7 = 0$
31. $x^2 - 4xy + 4y^2 - 5 = 0$
32. $2x^2 - 12xy + 18y^2 - 49 = 0$
33. $3x^2 + 5xy + 2y^2 - 8y - 1 = 0$
34 $2x^2 + 7xy + 9y^2 + 20x - 86 = 0$

MULTIPLE CHOICE QUESTIONS

1. **$\cosh x + \sinh x$ = ______.**
   (a) e^x (b) e^{-x} (c) 1 (d) -1
2. **$\cosh x - \sinh x$ = ______.**
   (a) 1 (b) e^x (c) e^{-x} (d) -1
3. **$\cosh x \cosh y + \sinh x \sinh y$ = ______.**
   (a) $\cosh(x - y)$ (b) $\sinh(x + y)$ (c) $\sinh(x - y)$ (d) $\cosh(x + y)$

4. **$\cosh x \cosh y - \sinh x \sinh y$ = ______.**
   (a) $\cosh(x + y)$ (b) $\cosh(x - y)$ (c) $\sinh(x - y)$ (d) $\sinh(x + y)$
5. **$\dfrac{d}{dx}(\operatorname{sech} x)$ = ______.**
   (a) $\operatorname{sech} x$ (b) $\tanh x$ (c) $\operatorname{sech} x \tanh x$ (d) $-\operatorname{sech} x \tanh x$
6. **$\dfrac{d}{dx}(\operatorname{cosech} x)$ = ______.**
   (a) $\operatorname{cosech} x$ (b) $\coth^2 x$ (c) $-\operatorname{cosech} x \coth x$ (d) $\operatorname{cosech} x \coth x$
7. **$\int \sinh x$ = ______ + c.**
   (a) $-\cosh x$ (b) $\cosh x$ (c) $h\cosh x$ (d) $-h\cosh x$
8. **$\int \operatorname{cosech}^2 x$ = ______ + c.**
   (a) $-\coth x$ (b) $\coth x$ (c) $\operatorname{cosech} x \coth x$ (d) $-\operatorname{cosech} x \coth x$
9. **$\int \operatorname{sech} x \tanh x \, dx$ = ______ + c.**
   (a) $\operatorname{sech} x$ (b) $\tan x$ (c) $-\operatorname{sech} x$ (d) $\tanh^2 x$
10. **$\int -\operatorname{cosech} x \coth x \, dx$ = ______ + c.**
    (a) $\operatorname{cosech} x$ (b) $-\operatorname{cosech} x$ (c) $\coth x$ (d) $-\coth^2 x$
11. **______ = $\ln(x + \sqrt{x^2 - 1})$, $x \geq 1$.**
    (a) $\sinh^{-1} x$ (b) $\cosh^{-1} x$ (c) $\cosh x$ (d) $\tanh^{-1} x$
12. **______ = $\dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right)$, $|x| > 1$.**
    (a) $\tanh^{-1} x$ (b) $\operatorname{sech}^{-1} x$ (c) $\coth^{-1} x$ (d) $\operatorname{cosech}^{-1} x$
13. **Domain of $\tanh x$ is ______.**
    (a) R (b) N (c) Z (d) R − {0}
14. **Domain of $\operatorname{cosech} x$ is ______.**
    (a) R (b) R − {0} (c) Z (d) N
15. **$\lim_{x \to 0} (\cot x)^{\sin 2x}$ is ______ form.**
    (a) $\dfrac{0}{0}$ (b) 1^∞ (c) ∞^0 (d) ∞^∞

16. $\lim_{x \to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$ is _______ form.

(a) $\frac{0}{0}$ (b) 1^{∞} (c) ∞^0 (d) 0^{∞}

17. $\lim_{x \to 1}(4-4x^2)^{\frac{1}{\log(2-2x)}}$ is _______ form.

(a) 1^{∞} (b) 1^0 (c) ∞^0 (d) 0^0

18. Directrix of $y^2 = 10x$ is _______.

(a) $x = -\frac{5}{2}$ (b) $x = \frac{5}{2}$ (c) $x = 10$ (d) $x = -10$

19. Foci of $\frac{x^2}{9} + \frac{y^2}{16} = 1$ are _______.

(a) $(0, \pm\sqrt{7})$ (b) $(\pm\sqrt{7}, 0)$ (c) $(0, \pm 4)$ (d) $(\pm 4, 0)$

20. Verties of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ are _______.

(a) $(\pm 3, 0)$ (b) $(\pm 2, 0)$ (c) $(0, \pm 3)$ (d) $(0, \pm 2)$

21. In $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the distance between center to focus is c = _______.

(a) $a^2 - b^2$ (b) $b^2 - a^2$ (c) $\sqrt{a^2 - b^2}$ (d) $\sqrt{b^2 - a^2}$

22. Asymptotes to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are y = _______.

(a) $\pm\frac{b}{a}x$ (b) $\pm\frac{a}{b}x$ (c) $\pm x$ (d) 0

23. Asymptotes to $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ are y = _______.

(a) $\pm\frac{b}{a}x$ (b) $\pm\frac{a}{b}x$ (c) $\pm x$ (d) 0

24. In $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ the distance between center to focus is c = _______.

(a) $a^2 + b^2$ (b) $a^2 - b^2$ (c) $\sqrt{a^2 + b^2}$ (d) $\sqrt{a^2 - b^2}$

25. The eccentricity of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is e = _______.

(a) $\frac{b^2 - a^2}{a}$ (b) $\sqrt{\frac{b^2 - a^2}{a}}$ (c) $\frac{\sqrt{a^2 - b^2}}{a}$ (d) $\sqrt{\frac{a^2 + b^2}{a}}$

26. The eccentricity of $12x^2 - 27y^2 = 108$ is e = _______.

(a) $\frac{\sqrt{5}}{3}$ (b) $\frac{5}{3}$ (c) $\frac{13}{3}$ (d) $\frac{\sqrt{13}}{3}$

27. $Ax^2 + Cy + Dx + Ey + F = 0$ represents a circle if _______.

(a) $A = C$ (b) $A \neq C$ (c) $A = C \neq 0$ (d) $A = -C$

28. $xy - y^2 - 5y + 1 = 0$ represents a _______.

(a) parabola (b) ellipse (c) circle (d) hyperbola

ANSWERS

1. (a),	2. (c),	3. (d),	4. (b),	5. (d),	6. (c),	7. (b),	8. (a),
9. (c),	10. (a),	11. (b),	12. (c),	13. (a),	14. (b),	15. (c),	16. (b),
17. (d),	18. (a),	19. (a),	20. (c),	21. (c),	22. (a),	23. (b),	24. (c),
25. (c),	26. (d),	27. (c),	28. (d).				

■ ✻ ■

UNIT 2

Curve sketching

■ **Definition :**

Cartesian equation : Equation in the form $f(x, y) = 0$ or $y = g(x)$ or $x = h(y)$ is called **Cartesian equation.**

Parametric equation : Equation in the form $x = f(t)$; $y = g(t)$ where t is parameter are called parametric equations.

e.g. $x^2 = 4ay$; $x^2 + y^2 = 4$; $y = \frac{x^2 - 1}{x^2 + 4}$; $y = \frac{(x-2)(x+1)}{x}$ are cartesian equations.

GRAPH OF CARTESIAN EQUATION

To sketch the graph of cartesian equation we have to discuss following points.

(1) Intercepts :

For

x – intercepts : put $y = 0$

y – intercepts : put $x = 0$

(2) Symmetry :

We have to discuss three types of symmetry.

(i) **Symmetry about X-axis :** Given curve is said to symmetry about X-axis, if we replace y by $-y$ equation remains unchanged.

(ii) **Symmetry about Y-axis :** Given curve is said to symmetry about Y–axis, if we replace x by $-x$ equation remains unchanged.

(iii) **Symmetry about origin :** Given curve is said to symmetry about origin, if we replace x by $-x$ and y by $-y$, equation remains unchanged.

■ **Remark :**

(i) If all powers of x are even powers, then equation is symmetry about Y-axis.

(ii) If all powers of y are even powers, then equation is symmetry about X-axis.

(iii) If given equation is symmetry about X-axis and Y-axis both then it is also symmetry about origin.

(3) Asymptote :

Discuss asymptote for Cartesian curve.

Ans. :

Def^n : A line is said to asymptote of the curve, if the perpendicular distance between points on the curve and points on the line tends to 0, when curve goes away from origin.

- There are two types of asymptotes.

(i) **Vertical asymptotes :** For the curve $y = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials. Take $q(x) = 0$ then find values of x.

(ii) **Horizontal asymptotes :** For the curve $y = \frac{p(x)}{q(x)} = \frac{a_0 + a_1x + a_2x^2 + ... + a_mx^m}{b_0 + b_1x + b_1x^2 + ... + b_nx^n}$.

→ If $m = n$ then $y = \frac{a_m}{b_n}$ is horizontal asymptote.

→ If $m < n$, then $y = 0$ is horizontal asymptote.

→ If $m > n$, then horizontal asymptote is not possible.

Ex. 1. Find asymptote for the curve $y = x^3 - 3x^2 - 2x$.

Sol^n. :

- **Vertical asymptote :** Take $q(x) = 0$

$\therefore$ $1 = 0$, not possible

$\therefore$ Vertical asymptote is not possible.

- **Horizontal asymptote :** $y = \frac{x^3 - 3x^2 - 2x}{1}$

$m = 3$, $n = 0$, $m > n$

$\therefore$ Horizontal asymptote is not possible.

(4) Sign of 'y' : For the curve $y = \frac{p(x)}{q(x)}$

- First take $p(x) = 0$ and $q(x) = 0$, then find values of x.
- Arrange above values in increasing order and make different intervals.
- Check sign of y in above intervals.

Ex. 2. Sketch the following curve. OR Trace the graph of following. OR Disscuss symmetry, intercepts, asymptotes and sign of function for the following curve. Hence sketch the curve.

(1) $y = x^3 - 3x^2 + 2x$ SPU, June-2012, December-2012

Sol^n. :

$y = x(x^2 - 3x + 2)$

$y = x(x-2)(x-1)$

(i) **Intercept :**

→ **X – intercept :** Put $y = 0$, we get.

$0 = x(x-2)(x-1)$

∴ $\boxed{x = 0, 1, 2}$ are x – int.

→ **Y – intercept :** Put $x = 0$, we get.

$\boxed{y = 0}$ is y – int.

(ii) **Symmetry :**

→ **Symmetry about X-axis :** If we replace y by $-y$, we get

$-y = x(x-2)(x-1)$

∴ $y = -x(x-2)(x-1)$

Thus equation is change.

∴ It is not symmetry about x-axis.

→ **Symmetry about Y-axis :** If we replace x by $-x$, we get

$y = -x(-x-2)(-x-1)$

∴ $y = -x(x+2)(x+1)$

Thus, equation is change.

∴ It is not symmetry about y–axis.

→ **Symmetry about origin :** If we replace x by $-x$ and y by $-y$. We get,

$-y = -x(-x-2)(-x-1)$

∴ $-y = -x(x+2)(x+1)$

∴ $y = x(x+2)(x+1)$

Thus equation is changed.

∴ It is not symmetry about origin.

(iii) **Asymptotes :**

Here, $y = \dfrac{x(x-2)(x-1)}{1}$

→ **Vertical Asymptotes :**

Take $1 = 0$, which is not possible, so vertical asymptote is not possible.

→ **Horizontal Asymptotes :**

Here, $m = 3$, $n = 0$, $m > n$.

∴ Horizontal asymptote is not possible.

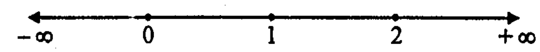
(iv) **Sign of 'y' :**

→ $y = x(x-2)(x-1)$

Take : $x(x-2)(x-1) = 0$

∴ $x = 0, 2, 1$

∴ $\boxed{x = 0, 1, 2}$



∴ Intervals are $(-\infty, 0)$, $(0, 1)$, $(1, 2)$, $(2, \infty)$

Here $y = x(x-2)(x-1)$

If $x \in (-\infty, 0)$, then

$y = (-ve)(-ve)(-ve) \Rightarrow y < 0$

If $x \in (0, 1)$, then

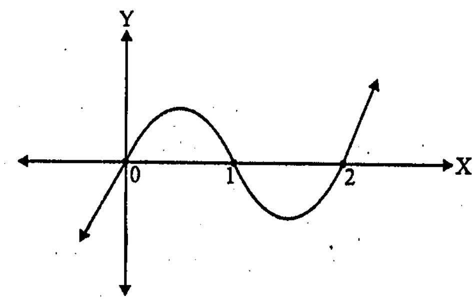
$y = (+ve)(-ve)(-ve) \Rightarrow y > 0$

If $x \in (1, 2)$, then

$y = (+ve)(-ve)(+ve) \Rightarrow y < 0$

If $x \in (2, \infty)$, then

$y = (+ve)(+ve)(+ve) \Rightarrow y > 0$



(2) $y = \dfrac{x^2 - 1}{x^2 - 4}$. **SPU, December-2014, June-2011, November-2010**

(i) **Intercepts :**

X–intercept : Put $y = 0$

∴ $0 = \dfrac{x^2 - 1}{x^2 - 4}$

∴ $0 = x^2 - 1$

∴ $\boxed{x = \pm 1}$ are x–int.

Y–intercept : Put $x = 0$

$y = \dfrac{-1}{-4} = \dfrac{1}{4}$

∴ $\boxed{y = 0.25}$ is y–int.

(ii) **Symmetry :**

→ **Symmetry about X-axis :** If we replace y by $-y$, we get $-y = \dfrac{x^2 - 1}{x^2 - 4}$ equation is changed.

∴ It is not symmetry about x-axis.

→ **Symmetry about Y-axis :** If we replace x by $-x$, we get $y = \dfrac{(-x)^2 - 1}{(-x)^4 - 4} = \dfrac{x^2 - 1}{x^2 - 4}$, equation is not changed.

∴ It is symmetry about y-axis.

→ **Symmetry about origin :** If we replace x by $-x$ and y by $-y$, we get

$$-y = \frac{(-x)^2 - 1}{(-x)^4 - 4}$$

$$-y = \frac{x^2 - 1}{x^4 - 4}$$ equation is changed.

∴ It is not symmetry about origine.

(iii) Asymptotes :

Here $y = \frac{x^2 - 1}{x^2 - 4}$

Vertical asymptote :

Take $x^2 - 4 = 0$ ∴ $x^2 = 4$ ∴ $\boxed{x = \pm 2}$ are vertical asymptotes.

Horizontal asymptote :

Here $m = 2$, $n = 2$, ∴ $m = n$

Then $\boxed{y = \frac{1}{1} = 1}$ is horizontal asymptote.

(iv) Sign of 'y' :

→ $y = \frac{x^2 - 1}{x^2 - 4}$

Take, $x^2 - 1 = 0$ and $x^2 - 4 = 0$

∴ $x^2 = 1$ ∴ $x^2 = 4$

∴ $x = \pm 1$ ∴ $x = \pm 2$

∴ $\boxed{x = -2, -1, 1, 2}$

∴ Intervals are $(-\infty, -2)$, $(-2, -1)$, $(-1, 1)$, $(1, 2)$, $(2, \infty)$

→ $y = \frac{x^2 - 1}{x^2 - 4}$

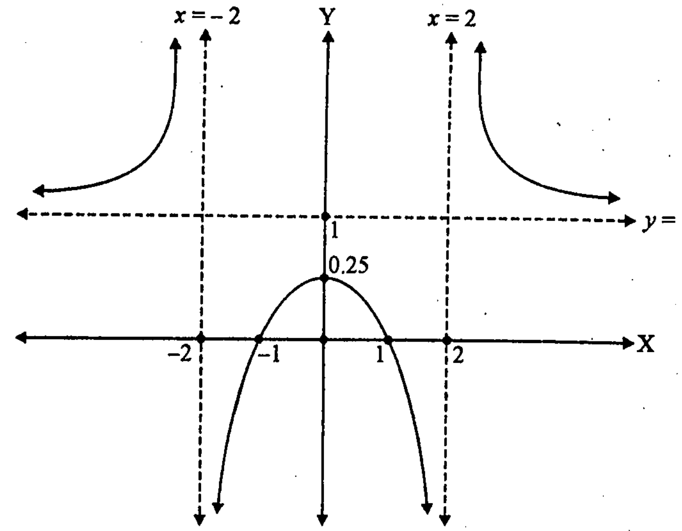
If $x \in (-\infty, -2)$, then $y = \frac{(+)}{(+)} \Rightarrow y > 0$

If $x \in (-2, -1)$, then $y = \frac{(+)}{(-)} \Rightarrow y < 0$

If $x \in (-1, 1)$, then $y = \frac{(-)}{(-)} \Rightarrow y > 0$

If $x \in (1, 2)$ then $y = \frac{(+)}{(-)} \Rightarrow y < 0$

If $x \in (2, \infty)$ then $y = \frac{(+)}{(+)} \Rightarrow y > 0$



(3) $y = \frac{2}{(x+1)(x-2)}$ or $y = \frac{2}{x^2 - x - 2}$ **SPU, Nov. 2015, April-2015**

Sol^n. :

(i) Intercept :

X-intercept : Put $y = 0$

∴ $0 = 2$ is not possible.

∴ X-intercept is not possible.

Y-intercept : Put $x = 0$

∴ $y = \frac{2}{1(-2)} \Rightarrow \boxed{y = -1}$ is Y-int.

(ii) Symmetry :

→ **Symmetry about X-axis :** If we replace y by $-y$, the equation is changed.

∴ It is not symmetry about X-axis.

→ **Symmetry about Y-axis :** If we replace x by $-x$, the equation is changed.

∴ It is not symmetry about Y-axis.

→ **Symmetry about origin :** If we replace x by $-x$ and y by $-y$, the equation is changed.

∴ It is not symmetry about origin.

(iii) Asymptotes :

$y = \dfrac{2}{x^2 - x - 2}$

Vertical asymptotes :

Take $x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$\therefore$ $\boxed{x = 2}$ or $\boxed{x = -1}$ are vertical asymptotes.

Horizontal asymptotes :

Here $m = 0$, $n = 2$ $\therefore$ $m < n$

$\therefore$ $\boxed{y = 0}$ is horizontal asymptote.

(iv) Sign of 'y' : $y = \dfrac{2}{x^2 - x - 2}$

$\rightarrow$ $x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$-\infty \quad -1 \quad 2 \quad +\infty$

$\therefore$ $\boxed{x = -1, 2}$

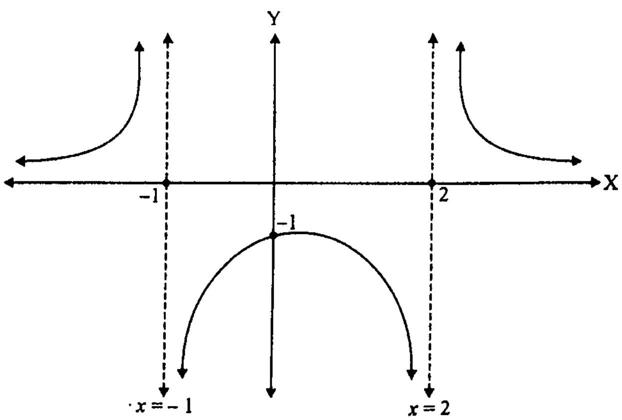
$\therefore$ Intervals are : $(-\infty, -1)$ $(-1, 2)$, $(2, \infty)$

Here $y = \dfrac{2}{(x-2)(x+1)}$

If $x \in (-\infty, -1)$, then $\dfrac{(+)}{(-)(-)}$ $\Rightarrow y > 0$

If $x \in (-1, 2)$, then $y = \dfrac{(+)}{(+)(-)}$ $\Rightarrow y < 0$

If $x \in (2, \infty)$, then $y = \dfrac{(+)}{(+)(+)}$ $\Rightarrow y > 0$



(4) $y = \dfrac{(x-1)(x+2)}{x(x-4)}$ or $y = \dfrac{x^2 + x - 2}{x^2 - 4x}$

SPU, April-2015, November-2012, June-2011, November-2010

Sol^n. :

(i) Intercept :

X-intercept : Put $y = 0$, we get $0 = (x - 1)(x + 2)$

$\therefore$ $\boxed{x = 1, -2}$ are X-intercepts :

Y-intercept : Put $x = 0$, we get $y = \dfrac{(-1)(2)}{0}$

$\therefore$ $\boxed{y = -\infty}$, not possible.

$\therefore$ Y-intercept is not possible.

(ii) Symmetry :

Symmetry about X-axis : If we replace y by $-y$, equation is changed.

$\therefore$ It is not symmetry about X-axis.

$\rightarrow$ **Symmetry about Y-axis :** If we replace x by $-x$ equation is changed.

$\therefore$ It is not symmetry about Y-axis.

$\rightarrow$ **Symmetry about origin :** If we replace x by $-x$ and y by $-y$ equation is changed.

$\therefore$ It is not symmetry about origin.

(iii) Asymptotes :

$\rightarrow$ Here, $y = \dfrac{(x-1)(x+2)}{x(x-4)}$

Vertical asymptote :

Take $x(x - 4) = 0$

$\therefore$ $\boxed{x = 0, 4}$ are vertical asymptotes.

Horizontal asymptote :

Here $m = 2$, $n = 2$ $\therefore$ $m = n$

$\therefore$ $y = \dfrac{1}{1} = \boxed{1}$ is horizontal asymptote.

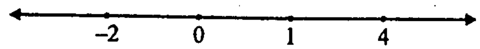
(iv) **Sign of 'y' :**

Here $y = \dfrac{(x-1)(x+2)}{x(x-4)}$

Take $(x-1)(x+2) = 0$ and $x(x-4) = 0$

$\boxed{x = 1, -2}$ $\quad \therefore \quad \boxed{x = 0, 4}$

$\therefore \quad \boxed{x = -2, 0, 1, 4}$



$\therefore$ Intervals are $(-\infty, -2)$, $(-2, 0)$, $(0, 1)$, $(1, 4)$, $(4, \infty)$.

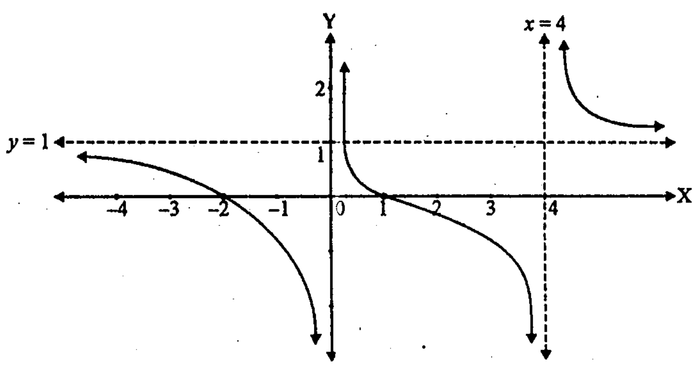
If $x \in (-\infty, -2)$, then $y = \dfrac{(-)(-)}{(-)(-)} \Rightarrow y > 0$

If $x \in (-2, 0)$, then $y = \dfrac{(-)(+)}{(-)(-)} \Rightarrow y < 0$

If $x \in (0, 1)$, then $y = \dfrac{(-)(+)}{(+)(-)} \Rightarrow y > 0$

If $x \in (1, 4)$, then $y = \dfrac{(+)(+)}{(+)(-)} \Rightarrow y < 0$

If $x \in (4, \infty)$, then $y = \dfrac{(+)(+)}{(+)(+)} \Rightarrow y > 0$



(5) $y = \dfrac{x(x-4)}{(x-1)(x+2)}$ **SPU, April-2016**

Sol^n. :

(i) **Intercept :**

X-int : Put $y = 0$, we get $0 = x(x-4)$

$\therefore \quad \boxed{x = 0, 4}$ are x–int.

Y-int : Put $x = 0$, we get $\boxed{y = 0}$ is y int.

(ii) **Symmetry :**

Symmetry about X-axis : If we repace y by $-y$, equation is changed.

$\therefore$ It is not symmetry about X-axis.

Symmetry about Y-axis : If we replace x by $-x$, equation is changed.

$\therefore$ It is not symmetry about Y-axis.

Symmetry about origin : If we replace x by $-x$ and y by $-y$, equation is changed.

$\therefore$ It is not symmetry about origin.

(iii) **Asymptets :**

Here, $y = \dfrac{(x-1)(x+2)}{x(x-4)}$

Vertical asymptotes : Take, $x(x-4) = 0 \quad \therefore \quad x = 0, 4$

$\therefore \quad \boxed{x = 0, 4}$ are vertical asymptotes.

Horizontal asymptotes : Here $m = 2$, $n = 2 \quad \therefore \quad m = n$

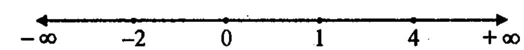
$\therefore \quad y = \dfrac{1}{1} = \boxed{1}$ is horizontal asymptote.

(iv) **Sign of 'y' :**

Take, $x(x-4) = 0$ and $(x-1)(x+2) = 0$

$\therefore \quad \boxed{x = 0, 4} \quad \therefore \quad \boxed{x = 1, -2}$

$\therefore \quad \boxed{x = -2, 0, 1, 4}$



$\therefore$ Intervals are $(-\infty, -2)$, $(-2, 0)$, $(0, 1)$, $(1, 4)$, $(4, \infty)$

Here $y = \dfrac{x(x-4)}{(x-1)(x+1)}$

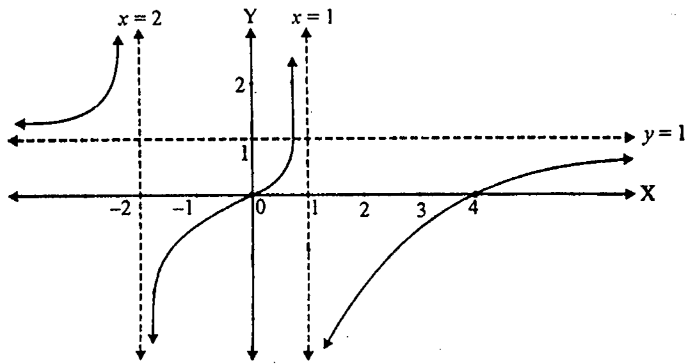
If $x \in (-\infty, -2)$, then $y = \dfrac{(-)(-)}{(-)(-)} \Rightarrow y > 0$

If $x \in (-2, 0)$, then $y = \dfrac{(-)(-)}{(-)(+)} \Rightarrow y < 0$

If $x \in (0, 1)$, then $y = \dfrac{(+)(-)}{(-)(+)} \Rightarrow y > 0$

If $x \in (1, 4)$, then $y = \dfrac{(+)(-)}{(+)(+)} \Rightarrow y < 0$

If $x \in (4, \infty)$, then $y = \dfrac{(+)(+)}{(+)(+)} \Rightarrow y > 0$



(6) $y = \dfrac{x^2 - 4}{x^2 - 1}$

Sol^n. :

(i) **Intercept :**

X-int : Put $y = 0$, we get $0 = x^2 - 4$

$\therefore \quad x^2 = 4 \quad \therefore \quad \boxed{x = \pm 2}$ are x–int.

Y-int : Put $x = 0$, we get $y = \dfrac{0-4}{0-1} \Rightarrow \boxed{y = 4}$ is y-int.

(ii) **Symmetry :**

Symmetry about X-axis : If we replace y by $-y$, equation is changed.

$\therefore$ It is not symmetry about X-axis.

Symmetry about Y-axis : If we replace x by $-x$, equation is not changed.

$\therefore$ It is symmetry about Y-axis.

Symmetry about origin : If we replace x by $-x$ and y by $-y$, equation is changed.

$\therefore$ It is not symmetry about origin.

(iii) **Asymptets :**

Here, $y = \dfrac{x^2 - 4}{x^2 - 1}$

Take $x^2 - 1 = 0$

$\therefore \quad x^2 = 1$

$\therefore \quad \boxed{x = -1, 1}$ are vertical asymptotes.

Also here $m = 2$, $n = 2 \quad \therefore \quad m = n$

$\therefore \quad y = 1$ is horizontal asymptote.

(iv) **Sign of 'y' :**

Take, $x^2 - 4 = 0$ and $x^2 - 1 = 0$

$x^2 = 4 \qquad x^2 = 1$

$\therefore \quad \boxed{x = \pm 2} \qquad \boxed{x = \pm 1}$

$\therefore \quad \boxed{x = -2, -1, 1, 2}$

$\therefore$ Intervals are $(-\infty, -2)$, $(-2, -1)$, $(-1, 1)$, $(1, 2)$, $(2, \infty)$.

Here $y = \dfrac{x^2 - 4}{x^2 - 1}$

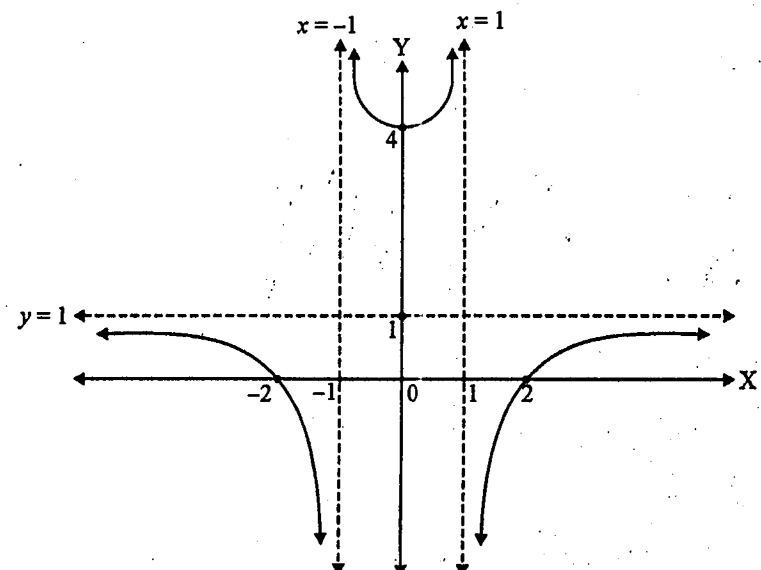
If $x \in (-\infty, -2)$, then $y = \dfrac{(+)}{(+)} \Rightarrow y > 0$

If $x \in (-2, -1)$, then $y = \dfrac{(-)}{(+)} \Rightarrow y < 0$

If $x \in (-1, 1)$, then $y = \dfrac{(-)}{(-)} \Rightarrow y > 0$

If $x \in (1, 2)$, the n $y = \dfrac{(-)}{(+)} \Rightarrow y < 0$

If $x \in (2, \infty)$, then $y = \dfrac{(+)}{(+)} \Rightarrow y > 0$



(7) $y = \frac{(x+3)(x-1)}{x(x+2)} = \frac{x^2+2x-3}{x^2+2x}$

Sol^n. :

(i) **Intercept :**

X-int : Put $y = 0$, we get $0 = (x+3)(x-1)$

$\therefore$ $\boxed{x = -3, 1}$ are x–int.

Y-int : Put $x = 0$, we get $y = \frac{(3)(-1)}{0}$

$\therefore$ $\boxed{y = \infty}$ not possible.

$\therefore$ y-intercept is not possible.

(ii) **Symmetry :**

Symmetry about X-axis : If we replace y by $-y$, equation is changed.

$\therefore$ It is not symmetry about X-axis.

Symmetry about Y-axis : If we replace x by $-x$, equation is changed.

$\therefore$ It is not symmetry about Y-axis.

Symmetry about origin : If we replace x by $-x$ and y by $-y$, equation is changed.

$\therefore$ It is not symmetry about origin.

(iii) **Asymptotes :**

Here, $y = \frac{(x+3)(x-1)}{x(x+2)}$

Take $x(x+2) = 0$

$\therefore$ $\boxed{x = 0, -2}$ are vertical asymptotes.

Also here $m = 2$, $n = 2$ $\quad\therefore\ m = n$

$\therefore$ $y = \frac{1}{1} \Rightarrow \boxed{y = 1}$ is horizontal asymptote.

(iv) **Sign of 'y' :**

Here, $y = \frac{(x+3)(x-1)}{x(x+2)}$

Take, $(x+3)(x-1) = 0$ and $x(x+2) = 0$

$\therefore$ $\boxed{x = -3, 1}$ $\qquad \therefore$ $\boxed{x = 0, -2}$

$\therefore$ $\boxed{x = -3, -2, 0, 1}$

$\therefore$ Intervals are $(-\infty, -3)$, $(-3, -2)$, $(-2, 0)$, $(0, 1)$, $(1, \infty)$

Here $y = \frac{(x+3)(x-1)}{x(x+2)}$

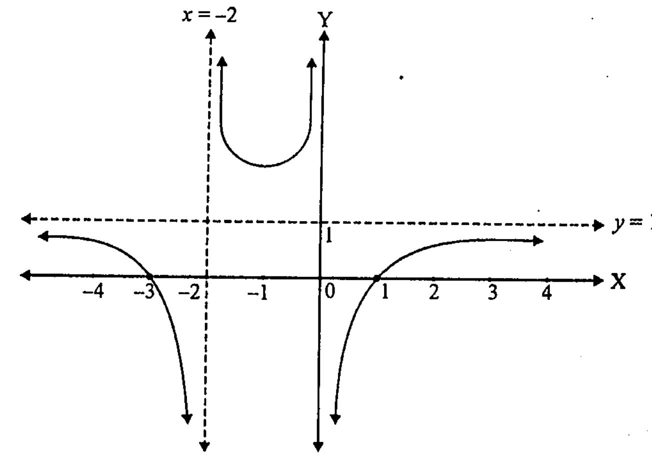
If $x \in (-\infty, -3)$, then $y = \frac{(-)(-)}{(-)(-)} \Rightarrow y > 0$

If $x \in (-3, -2)$, then $y = \frac{(+)(-)}{(-)(-)} \Rightarrow y < 0$

If $x \in (-2, 0)$, then $y = \frac{(+)(-)}{(-)(+)} \Rightarrow y > 0$

If $x \in (0, 1)$, then $y = \frac{(+)(-)}{(+)(+)} \Rightarrow y < 0$

If $x \in (1, \infty)$, then $y = \frac{(+)(+)}{(+)(+)} \Rightarrow y > 0$



(8) $y = \frac{x^3}{x^2 - 1}$

Sol^n. :

(i) **Intercept :**

X-int : Put $y = 0$, we get $0 = x^3$

$\therefore$ $\boxed{x = 0}$ are x–int.

Y-int : Put $x = 0$, we get

$y = \frac{0}{0-1} = \frac{0}{-1} = \boxed{0}$ is y-int.

(ii) **Symmetry :**

Symmetry about X-axis : If we replace y by $-y$, equation is changed.

$\therefore$ It is not symmetry about X-axis.

Symmetry about Y-axis : If we replace x by $-x$, equation is changed.

$\therefore$ It is not symmetry about Y-axis.

Symmetry about origin : If we replace x by $-x$ and y by $-y$, equation is not changed.

$\therefore$ It is symmetry about origin.

(iii) Asymptotes :

Here, $y = \dfrac{x^3}{x^2 - 1}$

Take $x^2 - 1 = 0$ $\quad\therefore\; x^2 = 1$

$\therefore$ $\boxed{x = \pm 1}$ are vertical asymptotes.

Also here $m = 3,\ n = 2$ $\quad\therefore\; m > n$

$\therefore$ Horizontal asymptote is not possible.

(iv) Sign of 'y' :

Take, $x^3 = 0$ and $x^2 - 1 = 0$

$\therefore\; \boxed{x = 0}$ $\quad\therefore\; \boxed{x = \pm 1}$

$\therefore\; \boxed{x = -1, 0, 1}$

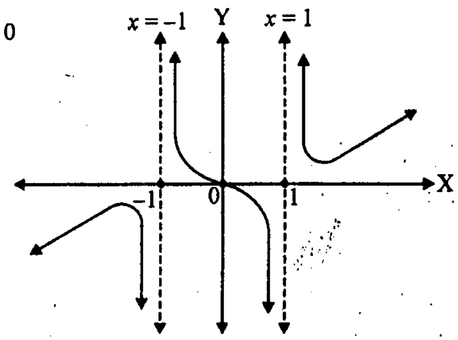
$\therefore$ Intervals are $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, $(1, \infty)$.

If $x \in (-\infty, -1)$, then $y = \dfrac{(-)}{(+)} \Rightarrow y < 0$

If $x \in (-1, 0)$, then $y = \dfrac{(-)}{(-)} \Rightarrow y > 0$

If $x \in (0, 1)$, then $y = \dfrac{(+)}{(-)} \Rightarrow y < 0$

If $x \in (1, \infty)$, then $y = \dfrac{(+)}{(+)} \Rightarrow y > 0$



(9) $y = \dfrac{(x-2)(x^2+1)}{(x-1)(x+1)^2}$

Sol^n. :

(i) Intercept :

X-int : Put $y = 0$,

$\therefore\; 0 = (x-2)(x^2+1)$

$\therefore\; x - 2 = 0$ or $x^2 + 1 = 0$

$\therefore\; x = 2$ or $x^2 = -1$ not possible.

$\therefore\; \boxed{x = 2}$ is x–intercept.

Y-int : Put $x = 0$

$\therefore\; y = \dfrac{(-2)(1)}{(-1)(1)} = \dfrac{-2}{-1}$

$\therefore\; \boxed{y = 2}$ is y-intercept.

(ii) Symmetry :

Symmetry about X-axis : If we replace y by $-y$, equation is changed.

$\therefore$ It is not symmetry about X-axis.

Symmetry about Y-axis : If we replace x by $-x$, equation is changed.

$\therefore$ It is not symmetry about Y-axis.

Symmetry about origin : If we replace x by $-x$ and y by $-y$, equation is changed.

$\therefore$ It is not symmetry about origin.

(iii) Asymptotes :

$y = \dfrac{(x-2)(x^2+1)}{(x-1)(x+1)^2}$

Take $(x-1)(x+1)^2 = 0$

$\Rightarrow\; x - 1 = 0,\ (x+1)^2 = 0 \;\Rightarrow\; x = 1, -1$

$\therefore\; \boxed{x = 1, -1}$ are vertical asymptotes.

Also here $m = 3,\ n = 3$ $\quad\therefore\; m = n$

$\therefore\; y = \dfrac{1}{1} \Rightarrow \boxed{y = 1}$ is horizontal asymptote.

(iv) Sign of 'y' :

Take, $(x-2)(x^2+1) = 0$ and $(x-1)(x+1)^2 = 0$

$\therefore\; \boxed{x = 2}$ $\quad\therefore\; \boxed{x = 1, -1}$

$\therefore\; \boxed{x = -1, 1, 2}$

$-\infty \quad -1 \quad 1 \quad 2 \quad +\infty$

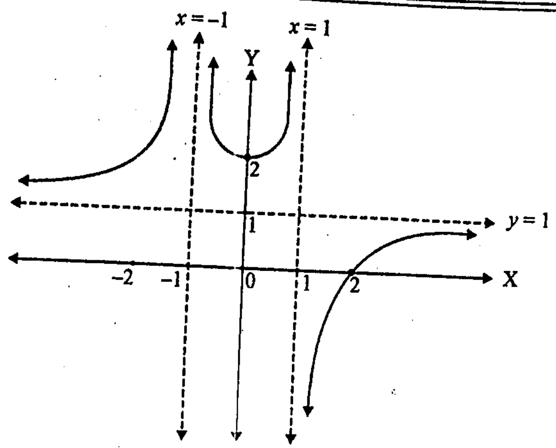
$\therefore$ Intervals are $(-\infty, -1)$, $(-1, 1)$, $(1, 2)$, $(2, \infty)$

If $x \in (-\infty, -1)$, then $y = \dfrac{(-)(+)}{(-)(+)} \Rightarrow y > 0$

If $x \in (-1, 1)$, then $y = \dfrac{(-)(+)}{(-)(+)} \Rightarrow y > 0$

If $x \in (1, 2)$, then $y = \dfrac{(-)(+)}{(+)(+)} \Rightarrow y < 0$

If $x \in (2, \infty)$, then $y = \dfrac{(+)(+)}{(+)(+)} \Rightarrow y > 0$



GRAPH OF PARAMETRIC EQUATION

To sketch the graph of parametric equation $x = f(t);\ y = g(t)$, we have to discuss following points.

1. **Intercept :** X-intercept and Y-intercept.
2. **Extent to the curve :** Extent to the curve is region of the curve on X-axis and Y-axis.
3. **Tangent parallel to the curve :** For the curve $x = f(t);\ y = g(t)$.

   We know that slope of tangent $= \frac{dy}{dx}$.

   (1) If $\left(\frac{dy}{dx}\right)_{p(x,\,y)} = 0$, then tangent at pt $p(x, y)$ is parallel to X-axis.

   (2) If $\left(\frac{dy}{dx}\right)_{p(x,\,y)} \longrightarrow \infty$ then tangent at pt $p(x, y)$ is parallel to Y-axis.
4. **Asymptotes to the curve :** For the curve $x = f(t),\ y = g(t)$. There are two type of asymptotes for parametric curves.

   (i) **Asymptotes parallel to axes :** Find limiting value of parameter of 't' for which any one variable x or y is finite and the other $\to \infty$. Finite value of x and y are called asymptotes parallel to axes.

   (ii) **Oblique asymptotes :** Find limiting value of parameter of 't' for which both variable $x \to \pm\infty$ and $y \to \pm\infty$, then there is a possibility of oblique asymptotes. If oblique asymptote is exists, then it is in the form $\boxed{y = mx + c}$ where

   $m = \lim\limits_{t} \frac{dy}{dx}$, $c = \lim\limits_{t} (y - mx)$.

Ex. 2. Find tangent parallel to axes for the following.

(1) $x = 2t^2;\ y = 3t$ **SPU, June-2011**

Sol^n. :

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \boxed{\frac{3}{4t}}$$

Here, $\frac{dy}{dx} = \frac{3}{4t} \neq 0,\ \forall\, t \in R$

$\therefore$ Tangent parallel to X-axis is not possible.

→ Also $\frac{dy}{dx} = \frac{3}{4t} \to \infty$, if $t = 0$

For $t = 0$, $x = 0$, $y = 0$

Thus we get, tangent parallel to Y-axis at point (0, 0).

(2) $x = \cos^2\theta;\ y = 2\sin\theta$ **SPU, April-2015, June-2012, December-2014, September-2014**

Sol^n. :

Here $x = \cos^2\theta;\ y = 2\sin\theta$

$\therefore \frac{dx}{d\theta} = -2\cos\theta\sin\theta$ and $\frac{dy}{d\theta} = 2\cos\theta$

$\therefore \frac{dy}{dx} = \frac{2\cos\theta}{-2\cos\theta\sin\theta} = \frac{-1}{\sin\theta}$

Here, $\frac{dy}{dx} = \frac{-1}{\sin\theta} \neq 0,\ \forall\, \theta \in R$

$\therefore$ Tangent parallel to X-axis is not possible.

Also $\frac{dy}{dx} = \frac{-1}{\sin\theta} \to \infty$ if $\theta \to n\pi,\ n \in Z$

For $\theta = n\pi$, $x = \cos^2 n\pi = 1$; $y = 2\sin n\pi = 0$

Thus we get, tangent parallel to Y-axis at point (1.0).

(3) $x = 4t^2 - 4t,\ y = 1 - 4t^2$

Sol^n. :

Here $\frac{dx}{dt} = 8t - 4$

$\frac{dy}{dt} = -8t$

$\therefore \quad \frac{dy}{dx} = \frac{-8t}{8t-4} = \frac{-2t}{2t-1} = \frac{2t}{1-2t}$

Clearly $\frac{dy}{dx} = \frac{2t}{1-2t} \to 0$, if $t = 0$.

For $t = 0$, $x = 0$, $y = 0$

$\therefore$ Tangent parallel to X-axis at (0, 0).

Also,

$\frac{dy}{dx} \to \infty$, if $1 - 2t = 0$, i.e. if $t = \frac{1}{2}$

For $t = \frac{1}{2}$, $x = 4 \cdot \frac{1}{4} - 4 \cdot \frac{1}{2} = 1 - 2 = -1$.

And $y = 1 - 4 \cdot \frac{1}{4} = 1 - 1 = 0$

Thus we get tangent parallel to Y-axis at point (– 1, 0).

Ex. 3 : Find extent of following.

(1) $x = 2t^2$, $y = 3t$, $t \in R$. SPU, June-2011

→ We know that,

$t \in R$	$t \in R$
$\therefore \ t^2 \geq 0$	$\therefore \ 3t \in R$
$\therefore \ 2t^2 \geq 0$	$\therefore \ y \in R$
$\therefore \ \boxed{x \geq 0}$	$\therefore \ \boxed{y \in (-\infty, \infty)}$

(2) $x = \cos^2\theta$, $y = 2\sin\theta$, $\theta \in R$

→ We know that

$-1 \leq \cos\theta \leq 1$	$-1 \leq \sin\theta \leq 1$
$\therefore \ 0 \leq \cos^2\theta \leq 1$	$\therefore \ -2 \leq 2\sin\theta \leq 2$
$\therefore \ \boxed{0 \leq x \leq 1}$	$\therefore \ \boxed{-2 \leq y \leq 2}$

Hence $0 \leq x \leq 1$, $-2 \leq y \leq 2$ are extents to the curve.

$\therefore$ Give curve is bounded.

■ **Theorem-1 :**

If a curve given by $x = f(t)$, $y = g(t)$ and both x and y get numerically large as t approches some number say a. Then an oblique asymptote to the curve if it exist is given by $y = mx + c$, where $m = \lim_{t \to a} \left(\frac{dy}{dx}\right)$, $c = \lim_{t \to a} (y - mx)$.

SPU, April-2016, Nov. 2015, April-2015, December-2014, September-2014, November-2013, June-2012, November-2012, December-2012, June-2011

Proof :

We have given that,

If $t \to a$, then $x \to \infty$ and $y \to \infty$.

$\therefore$ By definition, there is a possibility of oblique asymptote.

If it exist then it is in the form $y = mx + c$.

Now we find m :

We know that asymptote becomes a tangent to the curve at infinity. We know that

m = slope of tangent to the curve.

= slope of asymptote at the infinity

$= \frac{dy}{dx}$ at infinity

$= \frac{dy}{dx}$ at $x \to \infty$, $y \to \infty$

$= \frac{dy}{dx}$ if $t \to a$

$\therefore \quad \boxed{m = \lim_{t \to a} \left(\frac{dy}{dx}\right)}$

→ **Now we find c :**

We know that perpendicular distance between any point $p(x, y)$ of the curve to the line $mx - y + c = 0$ is $\left|\frac{mx - y + c}{\sqrt{m^2 + 1}}\right|$

We know that distance

$\frac{|mx - y + c|}{\sqrt{m^2+1}} \to 0$ at infinity i.e., at $x \to \infty$ and $y \to \infty$

Thus $\frac{|mx - y + c|}{\sqrt{m^2+1}} \to 0$, if $t \to a$

Thus $\lim_{t \to a} \frac{|mx - y + c|}{\sqrt{m^2+1}} = 0$

$\Rightarrow \lim_{t \to a} |mx - y + c| = 0$

$\Rightarrow \lim_{t \to a} mx - y + c = 0$

$\Rightarrow \boxed{c = \lim_{t \to a} y - mx}$

Ex. 4 : Find asymptotes for the curve given by $x = t + \frac{1}{t^2}$; $y = t - \frac{1}{t^2}$.

SPU, November-2013, June-2012

Sol^n. :

Asymptote parallel to axes :

Here $x = t + \frac{1}{t^2}$, $y = t - \frac{1}{t^2}$, $t \in R$

Here, we can not find any value of 't' for which one variable x or y is finite and other is infinite.

$\therefore$ Asymptote parallel to axes are not possible.

→ **Oblique asymptotes :**

If $t \to 0$, then $x \to \infty$ and $y \to \infty$

Also, If $t \to \infty$, then $x \to \infty$ and $y \to \infty$

$\therefore$ There is a possibility of oblique asymptote.

If it exist then it is in the form $\boxed{y = mx + c}$ .... (1)

→ **For $t \to 0$:**

We know that $\frac{dx}{dt} = 1 - \frac{2}{t^3} = 1 - 2t^{-3}$ and $\frac{dy}{dt} = 1 + \frac{2}{t^3} = 1 + 2t^{-3}$

$$\therefore \quad \frac{dy}{dx} = \frac{1 + 2t^{-3}}{1 - 2t^{-3}} = \frac{t^3 + 2}{t^3 - 2}$$

$$\therefore \quad m = \lim_{t \to 0} \frac{dy}{dx}$$

$$= \lim_{t \to 0} \left(\frac{t^3 + 2}{t^3 - 2}\right)$$

$$= \frac{0 + 2}{0 - 2}$$

$$\therefore \quad \boxed{m = -1}$$

Also, $c = \lim_{t \to 0} (y - mx)$

$$= \lim_{t \to 0} (y + x)$$

$$c = \lim_{t \to 0} \left(t - \frac{1}{t^2} + t + \frac{1}{t^2}\right)$$

$$= \lim_{t \to 0} (2t)$$

$$\therefore \quad \boxed{c = 0}$$

By equation (1)

$y = -x + 0$

$\therefore$ $\boxed{y = -x}$ is oblique asymptote.

→ **For $t \to \infty$:**

$$m = \lim_{t \to \infty} \frac{dy}{dx}$$

$$= \lim_{t \to \infty} \left(\frac{t^3 + 2}{t^3 - 2}\right)$$

$$= \lim_{t \to \infty} \left(\frac{1 + \frac{2}{t^3}}{1 - \frac{2}{t^3}}\right)$$

$$= \frac{1 + 0}{1 - 0} = 1$$

$$\therefore \quad \boxed{m = 1}$$

Also, $c = \lim_{t \to \infty} (y - mx) = \lim_{t \to \infty} (y - x)$

$$= \lim_{t \to \infty} \left(t - \frac{1}{t^2} - t - \frac{1}{t^2}\right)$$

$$= \lim_{t \to \infty} \frac{-2}{t^2}$$

$$= 0$$

$$\therefore \quad \boxed{c = 0}$$

$\Rightarrow$ $\boxed{y = x}$ is oblique asymptote.

Hence $\boxed{y = \pm x}$ are oblique asymptotes.

Ex. 5 : Obtain parametric equation of (i) Circle (ii) Ellipse (iii) Hyperbola (iv) Parabola.

SPU, November-2012

Sol^n :

(i) Circle :

We know that equation of circle with centre (0, 0) and radius 'r' is given by $x^2 + y^2 = r^2$

$$\Rightarrow \frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$\Rightarrow \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1 \quad \text{... (1)}$$

Let $\frac{x}{r} = \cos\theta$, $\frac{y}{r} = \sin\theta$, which satisfies above equation (1).

$\therefore$ $\boxed{x = r\cos\theta, y = r\sin\theta}$

Which is parametric equation of circle.

(ii) Ellipse :

We know that equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \text{.... (2)}$$

Let $\frac{x}{a} = \cos\theta$, $\frac{y}{b} = \sin\theta$ which satisfies above equation (2)

$\therefore$ $\boxed{x = a\cos\theta, y = b\sin\theta}$

Which is parametric equation of ellipse.

(iii) Hyperbola :

We know that equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\therefore \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1 \quad \text{..... (3)}$$

Let $\frac{x}{a} = \sec\theta$, $\frac{y}{b} = \tan\theta$ which satisfies above equation (3).

$\therefore$ $\boxed{x = a\sec\theta, y = b\tan\theta}$

which is parametric equation of hyperbola.

(iv) Parabola :

We know that equation of parabola is $y^2 = 4ax$.

Let $x = at^2$ then $y^2 = 4a(at^2)$

$= 4a^2t^2$

$\therefore$ $y = 2at$

Thus $\boxed{x = at^2, \quad y = 2at}$

Which is parametric equation of parabola.

Ex. 6 : Find parametric equation of following.

(1) $x^{2/3} + y^{2/3} = a^{2/3}$ SPU, April-2015, Dec.-2014, June-2011, Nov.-2010

$$\Rightarrow \left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1$$

$$\Rightarrow \left[\left(\frac{x}{a}\right)^{1/3}\right]^2 + \left[\left(\frac{y}{a}\right)^{1/3}\right]^2 = 1 \quad \text{.... (1)}$$

Let $\left(\frac{x}{a}\right)^{1/3} = \cos\theta$, $\left(\frac{y}{a}\right)^{1/3} = \sin\theta$, which satisfies equation (1)

Then $\boxed{x = a\cos^3\theta, y = a\sin^3\theta}$

(2) $\sqrt{x} + \sqrt{y} = \sqrt{a}$ SPU, Nov.-2015, Sept. 2014, Nov. 2013

$$\Rightarrow x^{1/2} + y^{1/2} = a^{1/2}$$

$$\therefore \left(\frac{x}{a}\right)^{1/2} + \left(\frac{y}{a}\right)^{1/2} = 1$$

$$\therefore \left[\left(\frac{x}{a}\right)^{1/4}\right]^2 + \left[\left(\frac{y}{a}\right)^{1/4}\right]^2 = 1 \quad \text{.... (2)}$$

Let, $\left(\frac{x}{a}\right)^{1/4} = \cos\theta$, $\left(\frac{y}{a}\right)^{1/4} = \sin\theta$, which satisfies equation (2).

$$\therefore \frac{x}{a} = \cos^4\theta, \quad \frac{y}{a} = \sin^4\theta$$

$\therefore$ $\boxed{x = a\cos^4\theta, \quad y = a\sin^4\theta}$

Ex. 7 : Sketch the curve given by following.

1. $x = a\sec\theta,\ y = b\tan\theta$

(1) Intercept :

x-intercept : put $y = 0$, we get

$0 = b\tan\theta$

$\therefore\ \tan\theta = 0$

$\therefore\ \dfrac{\sin\theta}{\cos\theta} = 0$

$\therefore\ \sin\theta = 0$

$\Rightarrow\ \theta = n\pi,\ n \in Z$

$\therefore\ x = a\sec(n\pi)$

$\therefore\ \boxed{x = \pm a}$ are x-intercepts.

y-intercept put : $x = 0$, we get

$0 = b\sec\theta$

$\therefore\ \sec\theta = 0$

$\therefore\ \dfrac{1}{\cos\theta} = 0$

$\therefore\ 1 = 0$, not possible

So, y-intercept is not possible

(2) Extent : we know that $-1 \le \cos\theta \le 1$;

$-1 \ge \sec\theta \ge 1$

$\sec\theta \ge 1$	$\sec\theta \le -1$
$\therefore\ a\sec\theta \ge a$	$a\sec\theta \le -a$
$\therefore\ \boxed{x \ge a}$	$\boxed{x \le -a}$

Also

$\tan\theta \in R$

$\therefore\ \boxed{y \in R}$

(3) Tangent parallel axes : $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{b\sec^2\theta}{a\sec\theta\tan\theta}$

$\therefore\ \dfrac{dy}{dx} = \dfrac{b}{a} \times \dfrac{1}{\cos\theta} \times \dfrac{\cos\theta}{\sin\theta} = \dfrac{b}{a\sin\theta}$

Tangent parallel to X-axis : we know that

$\dfrac{dy}{dx} = 0$, if $\dfrac{b}{a\sin\theta} = 0$,

if $\boxed{b = 0}$ which is not possible

So, tangent parallel to X-axis is not possible.

Tangent parallel to Y-axis : we know that

$\dfrac{dy}{dx} \to \infty$, if $\dfrac{b}{a\sin\theta} \to \infty$,

if $a\sin\theta = 0$,

if $\sin\theta = 0$,

if $\theta = n\pi,\ n \in Z$

$\therefore\ x = a\sec\theta$	$y = b\tan\theta$
$= a\sec(n\pi)$	$= b\tan(n\pi)$
$\therefore\ \boxed{x = \pm a}$	$\therefore\ \boxed{y = 0}$

So, we get tangent parallel to Y-axis at $(a, 0)$ and $(-a, 0)$.

(4) Asymptotes :

Asymptotes parallel to axes :

Here we can not find any limiting value of θ, for which one variable is finite and other is infinite. Therefore asymptotes parallel to axes are not possible.

Oblique asymptotoes :

If $\lim\limits_{t \to (2n+1)\frac{\pi}{2}}$ then $x \to \infty$ and $y \to \infty$. So there is apossibility of oblique asymptotes.

It is given by

$y = mx + c$

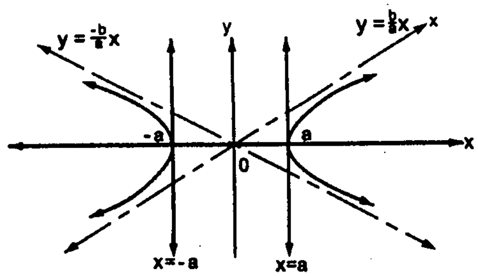
where, $m = \lim\limits_{t \to (2n+1)\frac{\pi}{2}} \left(\dfrac{dy}{dx}\right)$

$= \lim\limits_{t \to (2n+1)\frac{\pi}{2}} \left(\dfrac{b}{a\sin\theta}\right)$

$$= \frac{b}{a\sin(2n+1)\frac{\pi}{2}}$$

$$= \frac{b}{a(\pm 1)}$$

$\therefore$ $\boxed{m = \pm \frac{b}{a}}$



Also $c = \lim_{t \to (2n+1)\frac{\pi}{2}} (y - mx)$

$$= \lim_{t \to (2n+1)\frac{\pi}{2}} \left(b\tan\theta \pm \frac{b}{a} \cdot a\sec\theta\right)$$

$$= \lim_{t \to (2n+1)\frac{\pi}{2}} \frac{b(\sin\theta \mp 1)}{\cos\theta} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{t \to (2n+1)\frac{\pi}{2}} \frac{b\cos\theta}{-\sin\theta} = 0$$

$\therefore$ $\boxed{c = 0}$

$\therefore$ $y = mx + c$

$\Rightarrow$ $\boxed{y = \pm \frac{b}{a} x}$ is oblique asymptotes.

2. $x = \cos^2\theta$, $y = 2\sin\theta$

(1) Intercept :

x-intercept : put $y = 0$, we get

$0 = 2\sin\theta$

$\therefore$ $\sin\theta = 0$

$\therefore$ $\theta = n\pi$, $n \in Z$

Now, $x = \cos^2\theta = \cos^2(n\pi)$

$\therefore$ $\boxed{x = 1}$

So, $x = 1$ is x-intercept.

y-intercept put : $x = 0$, we get

$0 = \cos^2\theta$

$\therefore$ $\theta = (2n+1)\frac{\pi}{2}$, $n \in Z$

So $y = 2\sin(2n+1)\frac{\pi}{2} = \pm 2$

$\Rightarrow$ $\boxed{y = \pm 2}$

So, $y = \pm 2$ are y-intercept.

(2) Extent : We know that

$-1 \le \cos\theta \le 1$	$-1 \le \sin\theta \le 1$
$\therefore$ $0 \le \cos^2\theta \le 1$	$\Rightarrow$ $-2 \le 2\sin\theta \le 2$
$\therefore$ $\boxed{0 \le x \le 1}$	$\therefore$ $\boxed{-2 \le y \le 2}$

(3) Tangent parallel to axes :

$$\frac{dy}{dx} = \frac{2\cos\theta}{-2\cos\theta\sin\theta} = \frac{-1}{\sin\theta}$$

Tangent parallel to x-axis : we know that

$\frac{dy}{dx} = 0$, if $\frac{1}{\sin\theta} = 0$, i.e. $\boxed{1 = 0}$ not possible

So, tangent parallel to x-axis is not possible.

Tangent paralle to y-axis : We know that

$\frac{dy}{dx} \to \infty$, if $\frac{1}{\sin\theta} \to \infty$,

if $\sin\theta = 0$, if $\theta = n\pi$, $n \in Z$

So $x = \cos^2\theta$	$y = 2\sin\theta$
$= \cos^2(n\pi)$	$= 2\sin(n\pi)$
$\therefore$ $\boxed{x = 1}$	$\therefore$ $\boxed{y = 0}$

So, we get tangnet parallel to y-axis at (1, 0).

(4) Asymptotes :

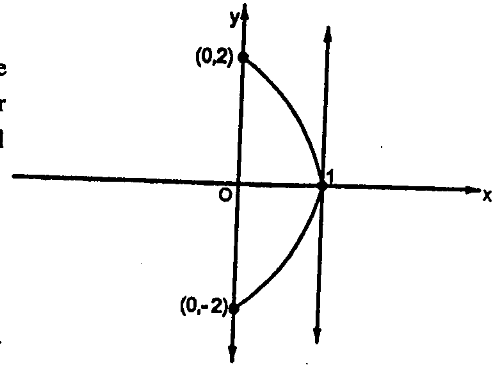
Asymptotes parallel to axes :

Here we can not find any limiting value of θ, for which one variable is finite and other is infinite. Therefore any asymptotes parallel to axes are not possible.

Oblique asymptotoes :

There does not axist any value of θ, for which both variable $x \to \infty$ and $y \to \infty$

$\therefore$ oblique asymptotes is not possible.



3. $x = 4t^2 - 4t, \quad y = 1 - 4t^2$ **:**

(1) Intercept :

x-intercept : put y = 0, we get

$0 = 1 - 4t^2$

$\therefore \quad 4t^2 = 1$

$\therefore \quad t^2 = \frac{1}{4}$

$\therefore \quad t = \pm\frac{1}{2}$

$x = 4\left(\frac{1}{4}\right) - 4\left(\frac{1}{2}\right),$	$x = 4\left(\frac{1}{4}\right) - 4\left(-\frac{1}{2}\right)$
$= 1 - 2$	$= 1 + 2$
$\therefore \quad \boxed{x = -1}$	$\therefore \quad \boxed{x = 3}$

So, $x = -1$ and $x = 3$ are x–intercepts.

y-intercept : Put $x = 0$, we get,

$0 = 4t^2 - 4t$

$\therefore \quad 0 = 4t(t - 1)$

$\therefore \quad \boxed{t = 0}$ or $\boxed{t = 1}$

$y = 1 - 0$	$y = 1 - 4$
$\therefore \quad \boxed{y = 1}$	$\boxed{y = -3}$

So, $y = 1$ and $y = -3$ are y-intercepts.

(2) Extent : We know that

$(2t - 1)^2 \ge 0$	$t^2 \ge 0$
$\Rightarrow \quad 4t^2 - 4t + 1 \ge 0$	$\Rightarrow \quad -4t^2 \le 0$
$\Rightarrow \quad 4t^2 - 4t \ge -1$	$\Rightarrow \quad 1 - 4t^2 \le 1$
$\Rightarrow \quad \boxed{x \ge -1}$	$\Rightarrow \quad \boxed{y \le 1}$

(3) Tangent parallel to axes :

$$\frac{dy}{dx} = \frac{-8t}{8t - 4} = \frac{-2t}{2t - 1}$$

Tangent parallel to x-axis : We know that

$$\frac{dy}{dx} = 0 \text{ if } \frac{-2t}{2t - 1} = 0,$$

if $-2t = 0$,

if $t = 0$

So, $x = 4t^2 - 4t$	$y = 1 - 4t^2$
$\therefore \quad \boxed{x = 0}$	$\therefore \quad \boxed{y = 1}$

So, we get tangent parallel to x-axis at (0, 1.).

Tangent parallel to y-axis : We know that

$$\frac{dy}{dx} \to \infty, \quad \text{if } \frac{-2t}{2t - 1} \to \infty,$$

if $2t - 1 = 0$,

if $2t = 1$,

if $t = \frac{1}{2}$

So, $x = 4\left(\frac{1}{4}\right) - 4\left(\frac{1}{2}\right)$	$y = 1 - 4\left(\frac{1}{4}\right)$
$= 1 - 2$	$= 1 - 1$
$\therefore \quad \boxed{x = -1}$	$\therefore \quad \boxed{y = 0}$

So, we get tangent parallel to y-axis at $(-1, 0)$.

(4) Asymptotes :

Asymptotes parallel to axies :

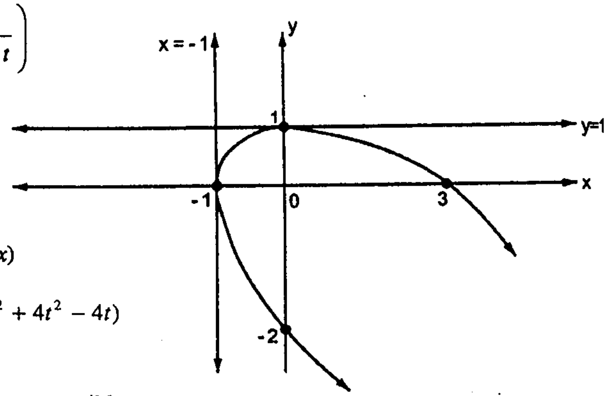
Here we can not find any limiting value of t, for which one variable is finite and other is infinite. Therefore asymptotes parallel to axes are not possible.

Oblique asymptotoes :

If $t \to \infty$, then $x \to \infty$ and $y \to \infty$. So there is apossibility of oblique asymptotes which is given by

$$y = mx + c$$

where,
$$m = \lim_{t \to \infty} \left(\frac{dy}{dx}\right)$$
$$= \lim_{t \to \infty} \left(\frac{-2t}{2t-1}\right)$$
$$= \lim_{t \to \infty} \left(\frac{-2t}{t(2-1/t)}\right)$$
$$= \lim_{t \to \infty} \left(\frac{-2}{2-1/t}\right)$$
$$= \frac{-2}{2-0}$$
$$\boxed{m = -1}$$



Also,
$$c = \lim_{t \to \infty} (y - mx)$$
$$= \lim_{t \to \infty} (1 - 4t^2 + 4t^2 - 4t)$$
$$= 1 - \infty$$
$\boxed{c = \infty}$ which is not possible.

$\therefore$ So, oblique asymptotes is not possible.

4. $x = t^2 + 4t, \quad y = t^2 + 3t$

(1) Intercept :

x-intercept : Put $y = 0$, we get,

$$0 = t^2 + 3t$$
$$0 = t(t+3)$$

$\therefore \ t = 0$ or $t = -3$

$\therefore \ \boxed{x = 0}$ | $\therefore \ x = 9 + 4(-3)$, $\boxed{x = -3}$

So, $x = 0$ and $x = -3$ are x-intercepts.

y-intercept : put $x = 0$, we get,

$$0 = t^2 + 4t$$
$$\therefore \ 0 = t(t+4)$$

$\therefore \ t = 0$ or $t = -4$

$\therefore \ \boxed{y = 0}$ | $\therefore \ y = 16 + 3(-4)$, $\therefore \ \boxed{y = 4}$

So, $y = 0$ and $y = 4$ are y-intercepts.

(3) Extent : We know that

$(t+2)^2 \geq 0$ | $\left(t + \frac{3}{2}\right)^2 \geq 0$

$\therefore \ t^2 + 4t + 4 \geq 0$ | $\therefore \ t^2 + 3t + \frac{3}{4} \geq 0$

$\therefore \ t^2 + 4t \geq -4$ | $\therefore \ t^2 + 3t \geq -\frac{3}{4}$

$\therefore \ \boxed{x \geq -4}$ | $\therefore \ \boxed{y \geq -\frac{3}{4}}$

(3) Tangent parallel to axes :

$$\frac{dy}{dx} = \frac{2t+3}{2t+4}$$

Tangent parallel to x-axis : We know that

$$\frac{dy}{dx} = 0, \text{ if } \frac{2t+3}{2t+4} = 0$$

if $2t + 3 = 0$,

if $2t = -3$,

if $t = -\frac{3}{2}$,

So, $x = \left(\frac{9}{4}\right) + 4\left(-\frac{3}{2}\right)$ | $y = \frac{9}{4} + 3\left(-\frac{3}{2}\right)$

$\therefore \ \boxed{x = -\frac{15}{4}}$ | $\therefore \ \boxed{y = -\frac{9}{4}}$

So, we get tangent parallel to x-axis at $\left(-\frac{15}{4}, -\frac{9}{4}\right)$.

Tangent parallel to y-axis : We know that

$$\frac{dy}{dx} \to \infty, \text{ if } \frac{2t+3}{2t+4} \to \infty$$

if $2t + 4 = 0$,

if $2t = -4$,

if $t = -2$

So $x = 4 + 4(-2)$ | $y = 4 + 3(-2)$

$\therefore$ $\boxed{x = -4}$ | $\therefore$ $\boxed{y = -2}$

So, we get tangent parallel to y-axis at $(-4, -2)$.

(4) Asymptotes :

Asymptotes parallel to axies :

Here we can not find any limiting value of t, for which one variable is finite and other is infinite. Therefore asymptotes parallel to axes are not possible.

Oblique asymptotoes :

If $t \to \infty$, then $x \to \infty$ and $y \to \infty$. So there is a possibility of oblique asymptotes which is given by

$$y = mx + c$$

where, $m = \lim_{t \to \infty} \left(\frac{dy}{dx}\right)$

$$= \lim_{t \to \infty} \left(\frac{2t+3}{2t+4}\right) = \lim_{t \to \infty} \frac{t\left[2 + \frac{3}{t}\right]}{t\left[2 + \frac{4}{t}\right]} = 1$$

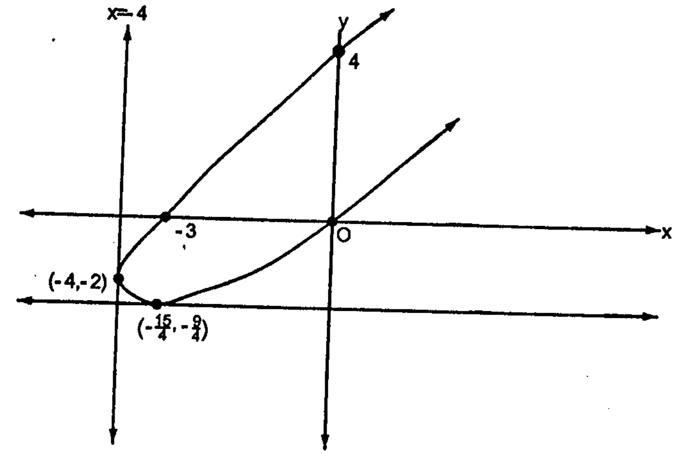
$\boxed{m = 1}$

Also, $c = \lim_{t \to \infty} (y - mx)$

$$= \lim_{t \to \infty} (t^2 + 3t - t^2 - 4t) = \lim_{t \to \infty} (-t)$$

$\boxed{c = -\infty}$ which is not possible.

So, oblique asymptotes is not possible.



5. $x = t + 1$, $y = \frac{3t}{t-4}$

(1) Intercept :

x-intercept : Put $y = 0$, we get,

$$0 = \frac{3t}{t-4}$$

$\therefore$ $3t = 0$

$\therefore$ $t = 0$

$\therefore$ $\boxed{x = 1}$

So, $x = 1$ is x-intercept.

y-intercept : put $x = 0$, we get,

$0 = t + 1$

$\therefore$ $t = 1$

$\therefore$ $y = \frac{-3}{-5}$

$\therefore$ $\boxed{y = \frac{3}{5}}$

So, $y = \frac{3}{5}$ is y-intercept.

(2) Extent :

$t \in R$ $\therefore$ $\boxed{x \in R}$ | $t \in R$ $\therefore$ $\boxed{y \in R}$

(3) Tangent parallel to axes :

$$\frac{dy}{dx} = \frac{3t - 12 - 3t}{(t-4)^2} = \frac{-12}{(t-4)^2}$$

Tangent parallel to x-axis : We know that

$$\frac{dy}{dx} \neq 0, \forall t \in R$$

So, tangent parallel to x-axis is not possible.

Tangent prallel to y-axis : We know that

$$\frac{dy}{dx} \to \infty, \text{ if } \frac{-12}{(t-4)^2} \to \infty,$$

if $(t - 4)^2 = 0$,

if $t - 4 = 0$

if $t = 4$,

So, $x = t + 1 = 4 + 1$ $\therefore$ $\boxed{x = 5}$ | $y = \frac{3t}{t-4} = \frac{12}{0}$ $\therefore$ $\boxed{y = \infty}$

So, tangent paralle to y-axis is not possible.

(4) Asymptotes :

Asymptotes parallel to axes : Clearly

If $t \to 4$, then $y \to \infty$ and $x \to 5$ so $x = 5$ is asymptotoe parallel to y-axis.

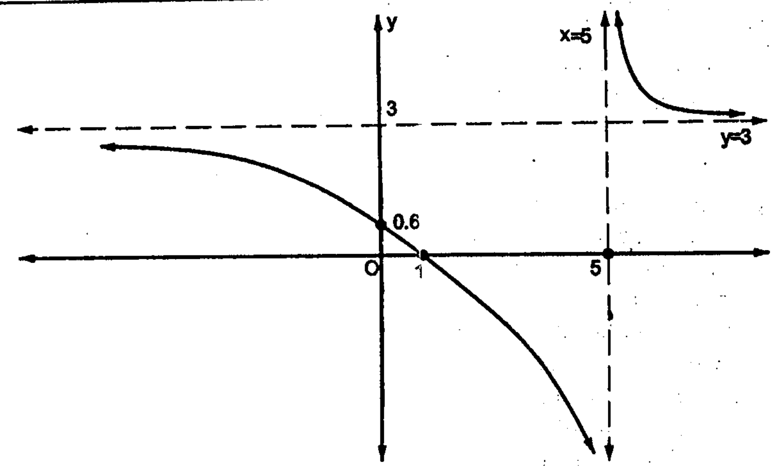
Also if $t \to \infty$, then $x \to \infty$ and $y \to 3$

$\therefore$ $y = 3$ is asymptote parallel to x-aixs.

Oblique asymptotes :

Here we can not find any limiting value of t, for which both $x \to \infty$ and $y \to \infty$.

$\therefore$ Oblique asymptote is not possible.



Ex. 8 : Express the following in parametric form using the given substitution. Then sketch the curve.

1. $x^2 - 2xy + y^2 + y = 0, x - y = t$

Solution : $x^2 - 2xy + y^2 + y = 0$

$\Rightarrow$ $(x-y)^2 + y = 0$

$\therefore$ $t^2 + y = 0$

$\therefore$ $\boxed{y = -t^2}$

Also, $x - y = t$

$\Rightarrow$ $x + t^2 = t$

$\Rightarrow$ $\boxed{x = t - t^2}$

(1) Intercepts :

x-intercept : put y = 0, we get $0 = t$

$\therefore$ $\boxed{x = 0}$

So, $x = 0$ is x-intercept.

y-intercept : put $x = 0$, we get

$0 = t(1 - t)$

$\therefore$ $t = 0$ or $t = 1$

$\therefore$ $\boxed{y = 0}$ | $\boxed{y = -1}$

So, $y = 0$ and $y = -1$ are y-intercepts.

(2) Extent : We know that

$$\left(t-\frac{1}{2}\right)^2 \geq 0$$

$$\therefore \quad -\left(t-\frac{1}{2}\right)^2 \leq 0$$

$$\Rightarrow \quad -\left(t^2 - t + \frac{1}{4}\right) \leq 0$$

$$\Rightarrow \quad -t^2 + t - \frac{1}{4} \leq 0$$

$$\therefore \quad t - t^2 \leq \frac{1}{4}$$

$$\therefore \quad \boxed{x \leq \frac{1}{4}}$$

$$t^2 \geq 0$$

$$\Rightarrow \quad -t^2 \leq 0$$

$$\therefore \quad \boxed{y \leq 0}$$

(3) Tangent parallel to axes :

$$\frac{dy}{dx} = \frac{-2t}{1-2t}$$

Tangent parallel to x-axis : We knowt hat

$$\frac{dy}{dx} = 0, \text{ if } 2t = 0 \text{ if } t = 0$$

So, $\boxed{x=0}$ and $\boxed{y=0}$

So, tangent parallel to x-axis at (0, 0)

Tangent parallel to y-axis : We know that

$$\frac{dy}{dx} \to \infty \text{ if } \frac{-2t}{1-2t} \to \infty,$$

if $1 - 2t = 0$,

if $2t = 1$,

if $t = \frac{1}{2}$,

So, $x = \frac{1}{2} - \frac{1}{4}$

$$\therefore \quad \boxed{x = \frac{1}{4}}$$

$$\boxed{y = -\frac{1}{4}}$$

So, we get tangent parallel to y-axis at $\left(\frac{1}{4}, -\frac{1}{4}\right)$.

(4) Asymptotes :

Asymptotes parallel to axes :

Here we can not find any limiting value of t, for which one variable is finite and other is infinite. Therefore asymptotes parallel to axes are not possible.

Oblique asymptotes :

Here if $x \to \infty$, then $y \to \infty$ and $y \to \infty$.

So, there is possibility of oblique asymptotes which is given by

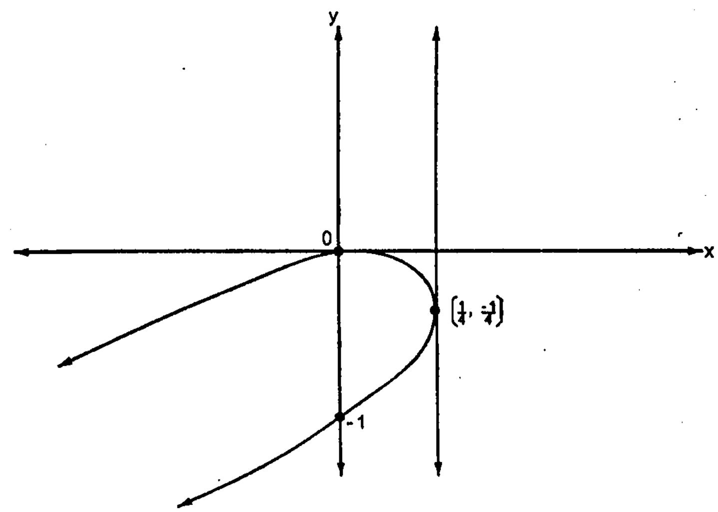
$$y = mx + c$$

where,
$$m = \lim_{t \to \infty} \left(\frac{dy}{dx}\right) = \lim_{t \to \infty} \left(\frac{-2t}{1-2t}\right) = \lim_{t \to \infty} \left(\frac{-2t}{t\,(1/t - 2)}\right)$$

$$\therefore \quad \boxed{m = 1}$$

Also
$$c = \lim_{t \to \infty} (y - mx) = \lim_{t \to \infty} (y - x) = \lim_{t \to \infty} (-t^2 - t + t^2)$$

$\boxed{c = \infty}$ which is not possible.



2. $x^2 + 2xy + y^2 + x = 0, \quad x + y = t$

Solution :

$x^2 + 2xy + y^2 + x = 0$

$\Rightarrow \quad (x+y)^2 + x = 0$

$\therefore \quad t^2 + x = 0$

$\therefore \quad \boxed{x = -t^2}$

Also, $x + y = t$

$\Rightarrow \quad -t^2 + y = t$

$\Rightarrow \quad y = t + t^2$

$\Rightarrow \quad \boxed{y = t\,(1+t)}$

(1) Intercepts :

x-intercept : put $y = 0$, we get $0 = t\,(1 + t)$

$\therefore \quad t = 0 \quad$ or $\quad t = -1$

$\therefore \quad \boxed{x = 0} \qquad \therefore \quad \boxed{x = -1}$

So, $x = 0$ and $x = -1$ are x-intercepts.

y-intercept : put $x = 0$ we get

$0 = -t^2 \qquad \therefore \quad t = 0$

$\therefore \quad \boxed{y = 0}$ is y-intercept

(2) Extent : We know that

$t^2 \geq 0 \qquad \left(t + \frac{1}{2}\right)^2 \geq 0$

$\therefore \quad -t^2 \leq 0 \qquad \therefore \quad t^2 + t + \frac{1}{4} \geq 0$

$\therefore \quad \boxed{x \leq 0} \qquad \therefore \quad \boxed{y \geq -\frac{1}{4}}$

(3) Tangent parallel to axes :

$$\frac{dy}{dx} = \frac{1+2t}{-2t}$$

Tangent parallel to x-axis : We know that

$$\frac{dy}{dx} = 0, \text{ if } \frac{1+2t}{-2t} = 0$$

if $1 + 2t = 0$,

if $2t = -1$,

if $t = -\frac{1}{2}$

So, $\boxed{x = -\frac{1}{4}}$ and $\boxed{y = -\frac{1}{4}}$

So, we get tangent parallel to x-axis at $\left(-\frac{1}{4}, -\frac{1}{4}\right)$

Tangent parallel to y-axis : We know that

$$\frac{dy}{dx} \to \infty, \text{ if } \frac{1+2t}{-2t} \to \infty, \text{ if } t = 0,$$

So, $\boxed{x = 0}$ and $\boxed{y = 0}$

So, tangent parallel to y-axis at (0, 0).

(4) Asymptotes :

Asymptotes parallel to axes :

Here we can not find any limiting value of t, for which one variable is finite and other is infinite. So asymptotes parallel to axes are not possible.

Oblique asymptotes :

If $t \to \pm\infty$, then $x \to \infty$ and $y \to \infty$.

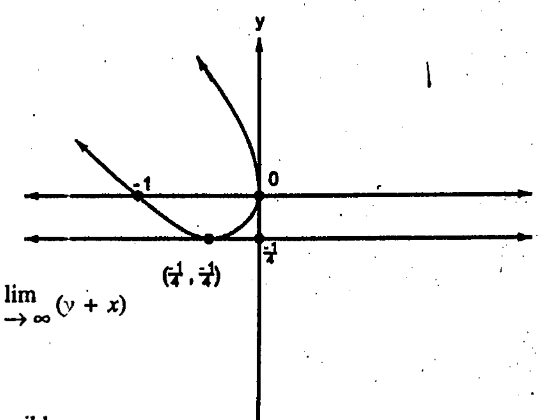
So, there is possibility of oblique asymptotes which is given by

$y = mx + c$, where

$$m = \lim_{t \to \infty} \left(\frac{dy}{dx}\right)$$

$$= \lim_{t \to \infty} \left(\frac{t\,(1/t + 2)}{-2t}\right)$$

$\therefore \quad \boxed{m = -1}$



Also $\quad c = \lim_{t \to \infty} (y - mx) = \lim_{t \to \infty} (y + x)$

$$= \lim_{t \to \infty} (t + t^2 - t^2)$$

$\boxed{c = \infty}$ which is not possible.

So, oblique asymptotes is not possible.

■ **Definition : Equation of tangent and normal :**

(1) **Equation of tangent** to the curve at point $p(x_1, y_1)$ is $y - y_1 = \left(\frac{dy}{dx}\right)_p (x - x_1)$.

(2) **Equation of normal** to the curve at point $p(x_1, y_1)$ is $y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_p}(x - x_1)$.

Ex. 9 : Find equation of tangent and normal to the parabola $y^2 = 4ax$ at the point t.

SPU, April-2016, June-2012, Dec. 2012

Sol^n. :

Given curve is $y^2 = 4ax$

→ Here, at 't' point means at point $p(at^2, 2at)$.

Now, $y^2 = 4ax$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_p = \frac{2a}{2at} = \boxed{\frac{1}{t}}$$

→ We know that equation of tangent to the parabola at the point $p(at^2, 2at)$;

$$y - y_1 = \left(\frac{dy}{dx}\right)_p (x - x_1)$$

$$\therefore \quad y - 2at = \frac{1}{t}(x - at^2)$$

$$\therefore \quad y - 2at = \frac{x}{t} - at$$

$$\therefore \quad \boxed{y = \frac{x}{t} + at}$$ which is equation of tangent.

→ We know that, equation of normal to the parabola at the point $p(at^2, 2at)$ is

$$y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_p}(x - x_1)$$

$$y - 2at = -\frac{1}{1/t}(x - at^2)$$

$$\therefore \quad y - 2at = -t(x - at^2)$$

$$\therefore \quad y - 2at = -xt + at^3$$

$$\therefore \quad \boxed{y = 2at - tx + at^3}$$ which is equation of normal.

Ex. 10 : Find equation of tangent and normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a\cos\theta, b\sin\theta)$.

Sol^n. :

SPU, November-2015, December-2014

Here given curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Differentiate with respect to x,

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{2y}{b^2}\frac{dy}{dx} = -\frac{2x}{a^2}$$

$$\therefore \quad \boxed{\frac{dy}{dx} = -\frac{xb^2}{ya^2}}$$

→ At pt. $p(a\cos\theta, b\sin\theta)$,

$$\left(\frac{dy}{dx}\right)_p = -\frac{a\cos\theta\, b^2}{b\sin\theta\, a^2} = -\frac{b\cos\theta}{a\sin\theta}$$

We know that equation of tangent to the ellipse at $p(a\cos\theta, b\sin\theta)$ is

$$y - y_1 = \left(\frac{dy}{dx}\right)_p (x - x_1)$$

$$\therefore \quad y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

$$\therefore \quad a\sin\theta y - ab\sin^2\theta = -b\cos\theta x + ab\cos^2\theta$$

$$\therefore \quad a\sin\theta y + b\cos\theta x = ab(\sin^2\theta + \cos^2\theta)$$

$$\therefore \quad b\cos\theta x + a\sin\theta y = ab$$

$$\therefore \quad \boxed{\frac{\cos\theta x}{a} + \frac{\sin\theta y}{b} = 1}$$ which is equation of tangent.

→ We know that equation of normal to the ellipse at $p(a\cos\theta, b\sin\theta)$ is

$$y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_p}(x - x_1)$$

$\therefore \quad y - b\sin\theta = \dfrac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$

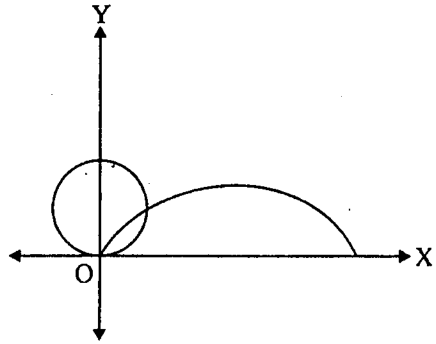
$\therefore \quad b\cos\theta y - b^2\sin\theta\cos\theta = a\sin\theta x - a^2\sin\theta\cos\theta$

$\therefore \quad b\cos\theta y - a\sin\theta x = (b^2 - a^2)\sin\theta\cos\theta$

$\therefore \quad \boxed{\dfrac{by}{\sin\theta} - \dfrac{ax}{\cos\theta} = b^2 - a^2}$ which is equation of normal.

Definition : Cycloid :

Cycloid is a path traced by a fixed point of a circle when circle rolls along a straight line without sliding.



Theorem-2 :

Obtain parametric equation of Cycloid.

OR Prove that a cycloid obtain by rolling a circle of radius 'a' is given by

$x = a(\theta - \sin\theta),\ \ y = a(1 - \cos\theta)$

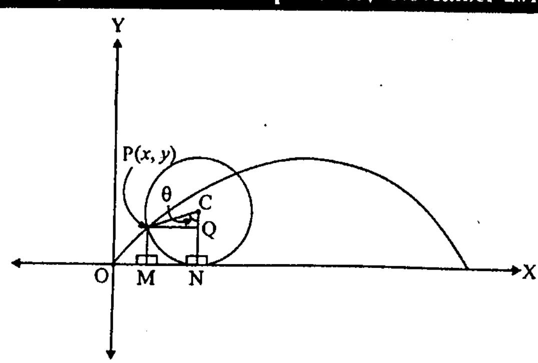
SPU, April-2016, November-2015 April-2015, November-2013

Proof :

If a circle with centre C and radius 'a' rolls along X-axis as shown in fig.

At starting, fixed point P is at origin 0. Let P(x, y) be any point on cycloid.

Rotate radius CP through angle θ as shown in fig.



Draw $\overline{CN}$ perpendicular to X-axis, $\overline{PM} \perp$ X-axis and $\overline{PQ} \perp \overline{CN}$, CN = PC = a.

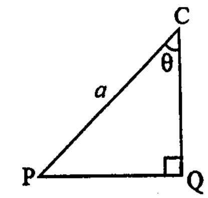
We know that arc length PN = $a\theta$ ($\because$ arc length = radius X angle.)

Also from right angle ΔPCQ,

$\cos\theta = \dfrac{CQ}{PC} = \dfrac{CQ}{a}$

$\therefore \quad \boxed{CQ = a\cos\theta}$

and $\sin\theta = \dfrac{PQ}{a}$ $\quad\therefore\quad \boxed{PQ = a\sin\theta}$



Also from fig. ON = arc length PN = $a\theta$

From fig.

x = OM
= ON − MN
= $a\theta$ − PQ
= $a\theta$ − $a\sin\theta$

$\therefore \quad \boxed{x = a(\theta - \sin\theta)}$

Also,

y = PM
= CN − CQ
= a − $a\cos\theta$

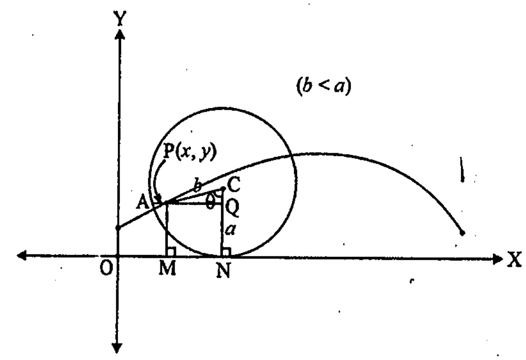
$\boxed{y = a(1 - \cos\theta)}$

Hence, $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ are required parametric equation of cycloid.

Ex. 11 : A circle of radius 'a' rolls along a line. Prove that the path traced by a point on the radius 'b' units ($b < a$) from the centre is given by $x = a\theta - b\sin\theta$; $y = a - b\cos\theta$.

Sol^n. :

If a circle with centre C and radius 'a' rolles along X-axis as shown in fig. At starting fixed point P is on Y-axis. Let P(x, y) be any point on cycloid. Rotate radius CA through angle 'θ'. Prove $\overline{CN} \perp$ X-axis, $\overline{PM} \perp$ X-axis and $\overline{PQ} \perp \overline{CN}$. Here CN = a, CP = b, CA = a.



We know that arc length AN = $a\theta$

($\because$ arc length = radius X angle)

From right angle ΔPQC, we get

$\cos\theta = \dfrac{CQ}{b}$

$\therefore$ CQ = $b\cos\theta$

$\sin\theta = \dfrac{PQ}{b}$

$\therefore$ PQ = $b\sin\theta$

→ From fig. ON = arc length AN = $a\theta$

From fig.

x = OM

x = ON − MN

= $a\theta$ − PQ

= $a\theta - b\sin\theta$

Also,

y = PM

= QN

= CN − CQ

∴ $y = a - b\cos\theta$

→ Hence, $x = a\theta - b\sin\theta$; $y = a - b\cos\theta$ are required equation.

Ex. 12 : A circle of radius 'a' rolls along X-axis prove that the path traced by a point on the radius b unit ($b > a$) from the centre is given by $x = a\theta - b\sin\theta$, $y = a - b\cos\theta$.

SPU, December-2014

Sol^n. :

If a circle with centre C and radius 'a' rolls along X-axis as shown in fig.

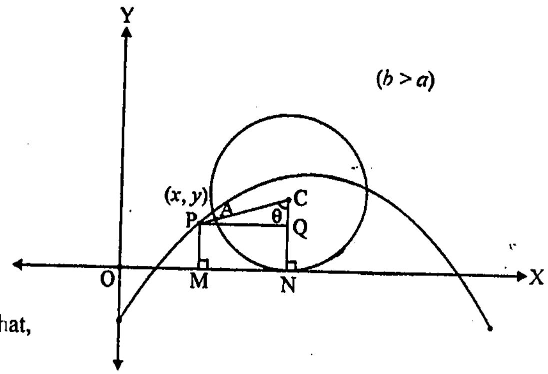
At starting pt. P is on Y-axis.

Let, P(x, y) be any pt. on cycloid.

Let rotate radius CA through angle 'θ'. Draw $\overline{CN} \perp$ X-axis, $\overline{PM} \perp$ X-axis. $\overline{PQ} \perp \overline{CN}$. Here CN = a, CP = b, CA = a.

We know that AN = $a\theta$.

(arc length = radius X angle)



From right angle ΔPQC, we say that,

$\cos\theta = \frac{CQ}{b}$, $\sin\theta = \frac{PQ}{b}$

∴ $\boxed{CQ = b\cos\theta}$ ∴ $\boxed{PQ = b\sin\theta}$

From figure ON = arc length AN = $a\theta$.

From right angle Δ PQC

x = OM

= ON − MN

= $a\theta$ − PQ

= $a\theta - b\sin\theta$

y = PM

= QN

= CN − CQ

= $a - b\cos\theta$

Hence, $x = a\theta - b\sin\theta$ and $y = a - b\cos\theta$ are required equation.

Ex. 13 : A circle of radius 3 units rolls along a line without sliding. Prove that the path traced by a pt. on the radius 2 units from the centre is given by $x = 3\theta - 2\sin\theta$, $y = 3 - 2\cos\theta$.

SPU, June-2011

Sol^n. :

Here, $a = 3$, $b = 2$

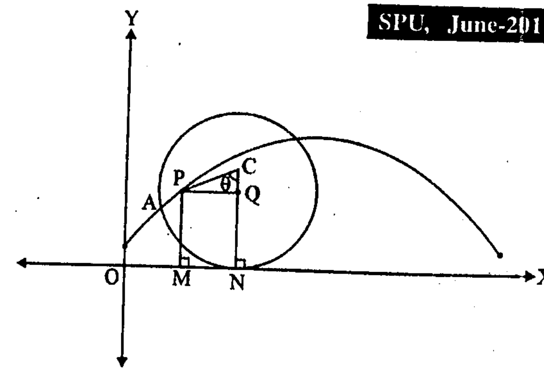
∴ $b < a$.

If a circle with radius a and centre C rolls along X-axis as shown in fig.

At starting fixed point P is an Y-axis.

Let P(x, y) be any point on cycloid.

Rotate radius CA through angle θ.



Draw $\overline{CN} \perp$ X-axis, $\overline{PM} \perp$ X-axis and $\overline{PQ} \perp \overline{CN}$. Here CN = a, CP = b, CA = a.

We know that, arc length AN = $a\theta$ (∵ arc length = radius X angle)

From right angle Δ PQC,

$\cos\theta = \frac{CQ}{b}$ $\sin\theta = \frac{PQ}{b}$

∴ $\boxed{CQ = b\cos\theta}$ ∴ $\boxed{PQ = b\sin\theta}$

From figure ON = arc length AN = $a\theta$

From figure

x = OM

= ON − MN

= AN − PQ

$\boxed{x = a\theta - b\sin\theta}$

y = PM

= CN − CQ

$\boxed{y = a - b\cos\theta}$

→ Hence, $x = a\theta - b\sin\theta$, $y = a - b\cos\theta$ are required equation.

Here, a = 3 and b = 2.

∴ $\boxed{x = 3\theta - 2\sin\theta}$, $\boxed{y = 3 - 2\cos\theta}$

Ex. 14 : Find asymptotes to the curve $y=\dfrac{2x^2-3}{x^2-4}$ **SPU, April-2015**

Sol^n. :

Vertical asymptotes : $x^2-4=0$

$\Rightarrow$ $\boxed{x=\pm 2}$

Horizontal asymptotes : Here $m=2$, $n=2$ $\quad\therefore\ m=n$

$\therefore\ y=\dfrac{2}{1}=2$ is horizontal asymptote.

Ex. 15 : Discuss all symmetries for $y=\dfrac{4-x^2}{x^2-9}$. **SPU, November-2013**

Sol^n. :

Symmetry about X-axis : If we replace y by $-y$ equation is changed.

$\therefore$ It is not symmetry about X-axis

Symmetry about Y-axis : If we replace x by $-x$, equation remains unchanged.

$\therefore$ It is symmetry about Y-axis.

Symmetry about origin : If we replace x by $-x$ and y by $-y$, equation is changed.

$\therefore$ If is not symmetry about origin.

Ex. 16 : Find equation of tangent to the curve given by $x=a\cos\theta$; $y=b\sin\theta$. **SPU, November-2013**

Sol^n. :

Here $x=a\cos\theta$; $y=b\sin\theta$

$\therefore\ \dfrac{dx}{d\theta}=-a\sin\theta;\ \dfrac{dy}{d\theta}=b\cos\theta$

$\therefore\ \dfrac{dy}{dx}=\dfrac{b\cos\theta}{-a\sin\theta}$

$\therefore$ Equation of tangent at point $(a\cos\theta, b\sin\theta)$ is $y-y_1=\left(\dfrac{dy}{dx}\right)(x-x_1)$.

$\Rightarrow\ y-b\sin\theta=\dfrac{-b\cos\theta}{a\sin\theta}(x-a\cos\theta)$

$\Rightarrow\ a\sin\theta y-ab\sin^2\theta=-b\cos\theta x+ab\cos^2\theta$

$\Rightarrow\ b\cos\theta x+a\sin\theta y=ab$

$\Rightarrow\ \boxed{\dfrac{\cos\theta x}{a}+\dfrac{\sin\theta y}{b}=1}$

Ex. 17 : Find horizontal and vertical asymptotes to the curve given by $x=3t+1$; $y=\dfrac{3t}{t-4}$. **SPU, June-2012**

Sol^n. :

If $t\to 4$ then $x=3\cdot 4+1=13$ and $y\to\infty$

$\therefore$ $\boxed{x=13}$ is vertical asymptote.

It $t\to\infty$, then $x\to\infty$ and $y=\lim_{t\to\infty}\dfrac{3t}{t\left(1-\dfrac{4}{t}\right)}=\dfrac{3}{1}=3$

$\therefore$ $\boxed{y=3}$ is horizontal asymptote.

Ex. 18 : Discuss symmetries of the curve $xy-16=0$. **SPU, June-2012**

Sol^n. :

Here $xy-16=0$

$\Rightarrow$ $\boxed{y=\dfrac{16}{x}}$

Symmetry about X-axis

$\therefore$ If we replace y by $-y$, equation is changed.

$\therefore$ It is not symmetry about X-axis

Symmetry about Y-axis. If we replace x by $-x$, equation is changed.

$\therefore$ It is not symmetry about Y-axis.

Symmetry about origin :

It we replace x by $-x$ and y by $-y$ then $-y=\dfrac{16}{-x}\ \Rightarrow\ y=\dfrac{16}{x}$.

Thus equation remains unchanged.

$\therefore$ It is symmetry about origin.

MULTIPLE CHOICE QUESTIONS

- **Fill in the Blanks :**

1. Asymptotes of $y = x^3 - 3x^2 + 2x$ are ____.
   (a) $x = 0, 1, 2; y = 1$ (b) $x = 0, -1, 2; y = 0$
   (c) $x = 0, 1, -2$ (d) Not possible

2. Asymptotes of $y = \frac{2}{(x+1)(x-2)}$ are ____.
   (a) $x = 1, -2; y = 1$ (b) $x = -1, 2; y = 0$
   (c) $x = 1, 2; y = 1$ (d) Not possible

3. Asymptotes of $y = \frac{(x-1)(x+2)}{x(x-4)}$ are ____.
   (a) $x = 0, 4; y = 1$ (b) $x = 1, -2; y = 1$
   (c) $x = 1, -2; y = 0$ (d) $x = 1, -2$

4. Asymptotes of $y = \frac{x^2-1}{x^2-4}$ are ____.
   (a) $x = 2, -2; y = 1$ (b) $x = 1, -1; y = 1$
   (c) $x = 2, -2; y = 0$ (d) $x = 1, -1, -2$

5. $y = x^3 - 3x^2 + 2x$ is symmetric about ____.
   (a) X-axis (b) Y-axis
   (c) Origin (d) None of these

6. $y = \frac{2}{3x}$ is symmetric about ____.
   (a) X-axis (b) Y-axis
   (c) Origin (d) None of these

7. $y = \frac{(x-1)(x+2)}{x(x-4)}$ is symmetric about ____.
   (a) X-axis (b) Y-axis
   (c) Origin (d) None of these

8. $y = \frac{x^2-1}{x^2-4}$ is symmetric about ____.
   (a) X-axis (b) Y-axis
   (c) Origin (d) None of these

9. Parametric equation for $x^{2/3} + y^{2/3} = a^{2/3}$ are ____.
   (a) $x = a\cos^3\theta; y = a\sin^3\theta$ (b) $x = a\cos^3\theta; y = b\sin^3\theta$
   (c) $x = \cos^3\theta; y = \sin^3\theta$ (d) $x = a\cos^4\theta; y = a\sin^4\theta$

10. Parametric equation for $\sqrt{x} + \sqrt{y} = \sqrt{a}$ are ____.
    (a) $x = a\cos^4\theta; y = b\sin^4\theta$ (b) $x = a\cos^3\theta; y = b\sin^3\theta$
    (c) $x = \cos^4\theta; y = \sin^4\theta$ (d) $x = a\cos^4\theta; y = a\sin^4\theta$

11. The curve of $y = x^3 - 3x^2 + 2x$ has ____ branches.
    (a) 1 (b) 2
    (c) 3 (d) 4

12. The curve of $y = \frac{2}{(x+1)(x-2)}$ has ____ branches.
    (a) 1 (b) 2
    (c) 3 (d) 4

13. The curve of $y = \frac{(x-1)(x+2)}{x(x-4)}$ has ____ branches.
    (a) 1 (b) 2
    (c) 3 (d) 4

14. The curve of $y = \frac{x^2-1}{x^2-4}$ has ____ branches.
    (a) 1 (b) 2
    (c) 3 (d) 4

15. The curve $y = \frac{4-x^2}{x^2-9}$, is symmetric with respect to ____.
    (a) Origin (b) X-axis
    (c) Line $y = x$ (d) Y-axis

16. Horizontal asymptote for the curve $xy - 3y - 9 = 0$ is ____.
    (a) $y = 0$ (b) $x = 0$
    (c) $y = 1$ (d) None

17. If $\left(\frac{dy}{dx}\right)_P = 0$ then the tangent at point P is parallel to the ____.
    (a) Y-axis (b) X-axis
    (c) Line $x = 5$ (d) None

18. Vertical asymptote for the curve $7xy = 2$ is ______.

(a) $y = 0$ (b) $x = 0$

(c) $y = 1$ (d) None

19. The curve $x = \frac{y^2 - 4}{y^2 - 9}$, is symmetric with respect to ______.

(a) Origin (b) X-axis

(c) Line $y = x$ (d) Y-axis

20. Vertical asymptote for the curve $x = \frac{5}{t}$, $y = t$ is ______.

(a) $y = 0$ (b) $x = 0$

(c) $y = 1$ (d) None

21. Parametric equation of an ellipse is ______.

(a) $x = a\cos\theta,\ y = b\sin\theta$ (b) $x = a\sec\theta,\ y = b\tan\theta$

(c) $x = \cos\theta,\ y = \sin\theta$ (d) None

22. Horizontal asymptote for the curve $x = \frac{1}{t}$, $y = 2t + 1$ is ______.

(a) $y = 0$ (b) $x = 0$

(c) $y = 1$ (d) None

23. Vertical asymptote for the curve $x = t + 1$, $y = \frac{3t}{t-4}$ is ______.

(a) $y = 0$ (b) $x = 5$

(c) $y = 1$ (d) None

24. Equation of a cycloid is ______.

(a) $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$ (b) $x = a(1 - \sin\theta),\ y = a(\theta - \cos\theta)$

(c) $x = a(\theta - \sin\theta)$ (d) $y = a(\theta - \cos\theta)$

ANSWERS

1. (d),	2. (b),	3. (a),	4. (a),	5. (d),	6. (c),	7. (d),	8. (b),
9. (a),	10. (d),	11. (a),	12. (c),	13. (c),	14. (c),	15. (d),	16. (a),
17. (b),	18. (b),	19. (b),	20. (b),	21. (a),	22. (c),	23. (b),	24. (a).

SHORT QUESTIONS

■ **Answer the following :**

1. Discuss intercepts, symmetry, horizontal and vertical asymptotes for each of the following curves :

(a) $y = \frac{1}{(x+1)^3}$ (e) $y = \frac{x^3}{x^2 - 1}$ (i) $y = \frac{4 - x^2}{x^2 - 9}$

(b) $y = \frac{2x}{x^2 - 9}$ (f) $y = \frac{x^2 - 1}{x^2 + 9}$ (j) $y = \frac{(x-1)(x-3)}{x(x-2)}$

(c) $y = \frac{x - 1}{x^2 + 9}$ (g) $y = \frac{x^2 + 4}{x^2 - 16}$ (k) $y = \frac{(2x-1)(x-3)^2}{(x-4)^2(x+1)}$

(d) $y = \frac{x^2}{x^2 - 16}$ (h) $y = \frac{x^2 - 9}{x^2 + 4}$ (l) $y = \frac{(x-2)(x^2+1)}{(x+2)(x+1)^2}$

2. Discuss intercepts, Extent, Tangent and asymptotes for each of the following curves.

(a) $x = 4\sec\theta;\ y = 3\tan\theta$ (f) $x = 3(\theta - \sin\theta);\ y = 3(\cos\theta - 1)$

(b) $x = 2\cos^2\theta;\ y = 3\sin\theta$ (g) $x = \frac{1}{t};\ y = t - \frac{1}{t}$

(c) $x = 4t^2 + 4t;\ y = 1 + 4t^2$ (h) $x = 1 - \cos t;\ y = 1 - \sin t$

(d) $x = t^2 + 4t;\ y = t^2 - 3t$ (i) $x = \sin\theta + \cos\theta;\ y = \sin\theta - \cos\theta$

(e) $x = t + 1;\ y = \frac{2t}{t+4}$ (j) $x = \cos 3\theta;\ y = 3\sin\theta$

3. Express the following in parametric form :

(a) $x^2 - 2xy + y^2 + y = 0$ (c) $(x + y)^3 - 2y(x + y)^2 - 8 = 0$

(b) $x^2 + 2xy + y^2 + x = 0$ (d) $(x + y)^3 - x(x + y)^2 + 1 = 0$

4. If curve is symmetry about both the axes then prove that it is symmetry about the origin.

EXERCISE

1. Discuss intercepts, symmetry, horizontal and vertical asymptotes and sign of function for each of the following curve. Hence sketch the curve.

OR

Sketch the curve given by following.

OR

Trace the curve given by following :

(a) $y = \dfrac{1}{(x-1)^3}$

(b) $y = \dfrac{2x}{x^2-4}$

(c) $y = \dfrac{x-1}{x^2-9}$

(d) $y = \dfrac{x^2}{x^2-1}$

(e) $y = \dfrac{x^3}{x^2+1}$

(e) $y = \dfrac{x^2-4}{x^2+4}$

(f) $y = \dfrac{x^2+4}{x^2-4}$

(g) $y = \dfrac{x^2+4}{x^2-4}$

(h) $y = \dfrac{x^2-9}{x^2-4}$

(i) $y = \dfrac{4-x^2}{x^2-9}$

(j) $y = \dfrac{(x+1)(x+2)}{x(x+4)}$

(k) $y = \dfrac{(x-1)(x+3)}{x(x+2)}$

(l) $y = \dfrac{(2x-1)(x+3)^2}{(x+4)(x-1)^2}$

(m) $y = \dfrac{(x+2)(x^2+1)}{(x-2)(x+1)^2}$

2. Sketch the curve given by following :

(a) $y = x^3 + 3x^2 + 2x$

(b) $y(x^2 - x - 2) = 2$

(c) $xy - 3x - y - 2 = 0$

(d) $3xy = 2$

(e) $xy - 3y - 9 = 0$

(f) $y = x(x^2 - 9)$

(g) $y = x^4 - 1$

(h) $y = (x+2)(x-1)^3$

(i) $y = (x-1)^2 (x+2)$

(j) $y = x^2 (x^2 - 9)$

(k) $y = x^3 (x^2 - 9)$

(l) $y = (x-1)^2 (x+2)^2$

(m) $x^2 y + 3y = 2x^2 + 7$

(n) $x^2 y + 4x = 4y$

(o) $xy - y - 2x = 0$

3. Sketch the curve given by following :

OR

Without converting into Cartesian form, sketch the graph of the curve given by following.

OR

Discuss intercepts, Extent, Tangent and asymptotes for each of the following curves. Hence sketch the curve.

(a) $x = 3\sec\theta;\ y = 2\tan\theta$

(b) $x = 2\cos^2\theta;\ y = 4\sin\theta$

(c) $x = 4t^2 + 4t;\ y = 1 + 4t^2$

(d) $x = t^2 - 4t;\ y = t^2 - 3t$

(e) $x = t + 1;\ y = \dfrac{t}{t+4}$

(f) $x = 2(\theta - \sin\theta);\ y = 2(\cos\theta - 1)$

(g) $x = \dfrac{1}{t};\ y = t - \dfrac{1}{t}$

(h) $x = 1 + \cos t;\ y = 1 - \sin t$

(i) $x = \sin\theta - \cos\theta;\ y = \sin\theta + \cos\theta$

(j) $x = \cos 2\theta;\ y = 2\sin\theta$

(k) $x = \dfrac{3t}{1+t^3};\ y = \dfrac{3t^2}{1+t^3}$

(l) $x = b\tan\theta;\ y = a\sec\theta$

(m) $x = a\,\mathrm{cosec}\,\theta;\ y = b\cot\theta$

(n) $x = b\cot\theta;\ y = a\,\mathrm{cosec}\,\theta$

4. Express the following in parametric form using the given substitution. Then sketch the curve.

(a) $x^2 - 2xy + y^2 + x = 0;\ x - y = t$

(b) $x^2 + 2xy + y^2 + y = 0;\ x + y = t$

(d) $(x+y)^3 - 2y(x+y)^2 - 8 = 0;\ x + y = 2t$

(c) $\sqrt{x} + \sqrt{y} = \sqrt{a};\ x = a\sin^4\theta$

(e) $(x+y)^3 - x(x+y)^2 + 1 = 0;\ x + y = t^{-1}$

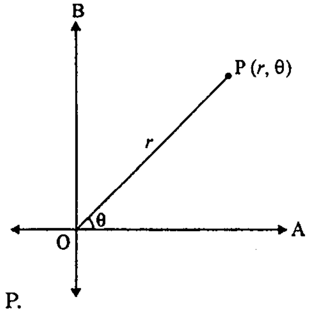
(f) $x^{2/3} + y^{2/3} = a^{2/3};\ x = a\cos^3\theta$

5. A circle of radius 2 rolls along a line without sliding. Show that the path traced by a point on the radius 3 units from the centre is given by $x = 2\theta - 3\sin\theta;\ y = 2 - 3\cos\theta$.

POLAR CURVES

■ **Definition :**

Polar equation : Let $f(x, y) = 0$ be a cartesian equation of a curve. By substituting $x = r\cos\theta$, $y = r\sin\theta$ in $f(x, y) = 0$, we get equation $g(r, \theta) = 0$ which is called polar equation of the curve.



■ **Polar Co-ordinates :**

In figure point 0 is called **origin** or **pole**.

→ $\overrightarrow{OA}$ is called **polar axis.**

→ $\overrightarrow{OB}$ is called **normal axis.**

→ Distance OP = r called **radius vector.**

→ Angle $\angle AOP = \theta$ is called **vectorial angle** of point P.

→ (r, θ) is called **polar co-ordinates** of point P.

■ **Remarks :**

(i) If angle θ measured anticlockwise direction from polar axis, then $\theta > 0$, otherwise $\theta < 0$.

(ii) Every point (r, θ), $-2\pi < \theta < 2\pi$ can be express three other ways.

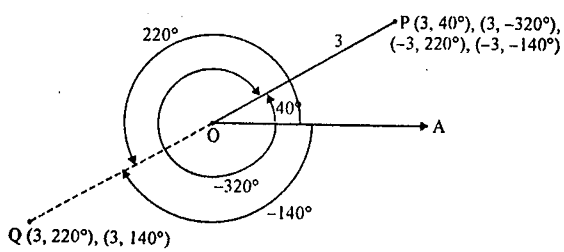
→ If $\theta \geq 0$, then $p(r, \theta)$ can be written as $(r, \theta - 360°)$, $(-r, \theta \pm 180°)$.

→ If $\theta < 0$, then $p(r, \theta)$ can be written as $(r, \theta + 360°)$, $(-r, \theta \pm 180°)$

Ex. 1. Express following point in the three other ways such that $-2\pi \leq \theta \leq 2\pi$.

(1) (3, 40°) **SPU, November-2013**

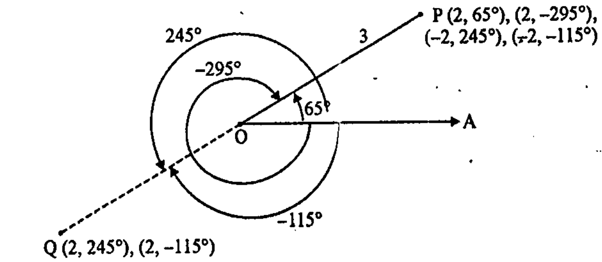
Sol^n. :



Thus (3, 40°) can written as (3, – 320°), (– 3, 220°), (– 3, – 140°)

(2) **(2, 65°)** **SPU, December-2014, June-2011**

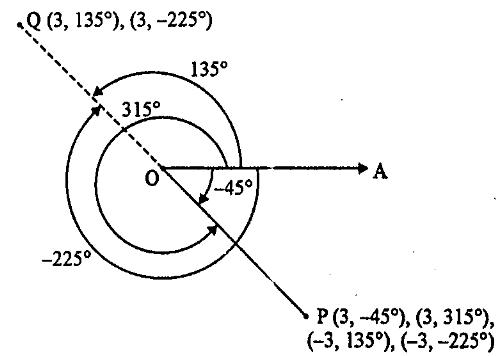
Sol^n. :



Thus (2, 65°) can be written as (2, – 295°), (– 2, 245°), (– 2, – 115°)

(3) **(3, – 45°)** **SPU, September-2014**

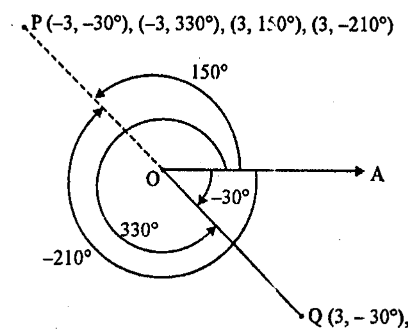
Sol^n. :



Thus (3, – 45°) can be written as (3, 315°), (– 3, 135°), (– 3, – 225°)

(4) **(– 3, – 30°)**

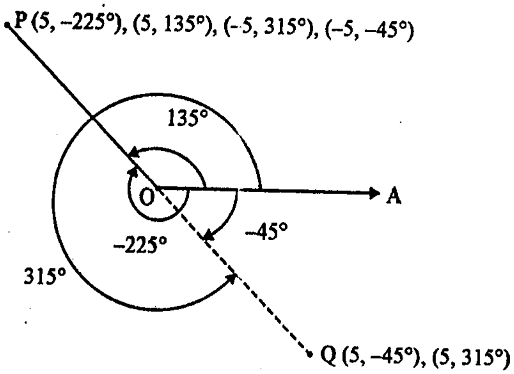
Sol^n. :



Thus point $p(-3, -30°)$ can be written as (– 3, 330°), (3, 150°), (3, – 210°)

(5) (5, – 225°)

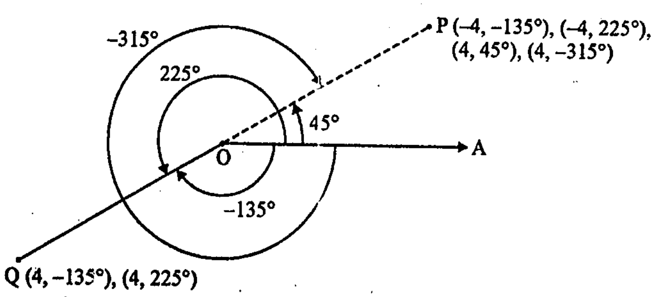
Sol^n. :



Thus point (5, – 225°) can be written as (5, 135°), (– 5, 315°) and (– 5, – 45°).

(6) (– 4, – 135°)

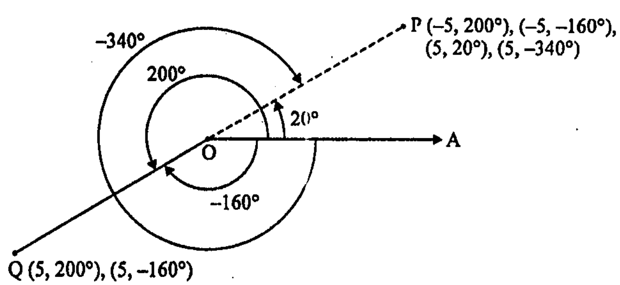
Sol^n. :



Thus point (– 4, – 135°) can be written as (– 4, 225°), (4, 45°) and (4, – 315°).

(7) (– 5, 200°)

Sol^n. :



Thus point (– 5, 200°) can be written as (– 5, – 160°), (5, 20°), (5, – 340°).

■ **Relation between Cartesian Co-ordinates and polar co-ordinates :**

SPU, June-2012

Let P(x, y) be any point, let (r, θ) be polar Co-ordinates of P. Draw $\overline{PM} \perp$ x-axis.

→ From right angle ΔOPM,

$$\cos\theta = \frac{x}{r}$$

$$\therefore \quad \boxed{x = r\cos\theta}$$

and $\sin\theta = \dfrac{y}{r}$

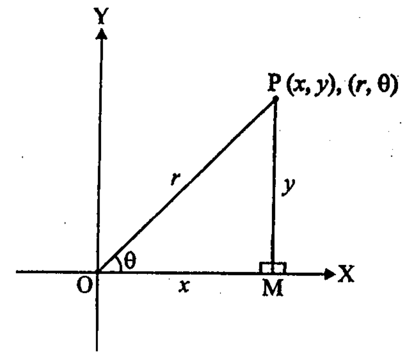
$$\therefore \quad \boxed{y = r\sin\theta}$$

Also $x^2 + y^2 = r^2$

$$\therefore \quad \boxed{r = \sqrt{x^2 + y^2}}$$

Also, $\dfrac{y}{x} = \tan\theta$

$$\therefore \quad \boxed{\theta = \tan^{-1}\frac{y}{x}}$$



Ex. 2. Express following point in polar form.

(1) $(\sqrt{3}, 1)$ **SPU, April-2015, November-2010, 2013**

Sol^n. :

Here, $x = \sqrt{3}$, $y = 1$

$\therefore \quad r = \sqrt{x^2 + y^2} = \sqrt{3+1} = \sqrt{4}$

$\therefore \quad r = 2$

Also $\tan\theta = \dfrac{y}{x} = \dfrac{1}{\sqrt{3}}$

$\theta = \tan^{-1}\dfrac{1}{\sqrt{3}}$

$\therefore \quad \theta = \dfrac{\pi}{6}$ or $\pi + \dfrac{\pi}{6}$

$\therefore \quad \boxed{\theta = \dfrac{\pi}{6}}$ [$\because$ $x > 0$, $y > 0$, point is in 1st quadrant]

Thus $(r, \theta) = \left(2, \dfrac{\pi}{6}\right)$ is required polar form.

(2) $(-\sqrt{3}, -1)$ **SPU, Nov. 2012, June-2011**

Sol^n. :

Here, $x = -\sqrt{3}$, $y = -1$

$\therefore \quad r = \sqrt{x^2 + y^2} = \sqrt{3+1} = \sqrt{4} = 2$

$\therefore \quad r = 2$

Also $\tan\theta = \dfrac{y}{x} = \dfrac{-1}{-\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

$\therefore \quad \theta = \tan^{-1}\dfrac{1}{\sqrt{3}}$

$\therefore \quad \theta = \dfrac{\pi}{6}$ or $\pi + \dfrac{\pi}{6}$

$\Rightarrow \quad \theta = \pi + \dfrac{\pi}{6}$ $\quad [\therefore\ x < 0, y < 0$, point is in 3rd quadrant$]$

Thus $(r, \theta) = \left(2, \dfrac{7\pi}{6}\right)$ is required polar form.

(3) $(-\sqrt{3}, 1)$ **SPU, June-2012, Dec. 2014**

Sol^n. :

Here $x = -\sqrt{3}$, $y = 1$

$r = \sqrt{x^2 + y^2} = \sqrt{3+1} = \sqrt{4}$

$\therefore \quad r = 2.$

Also $\tan\theta = \dfrac{y}{x} = \dfrac{1}{-\sqrt{3}}$

$\therefore \quad \theta = \tan^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)$

$\theta = \pi - \dfrac{\pi}{6}$ or $-\dfrac{\pi}{6}$

$\theta = \pi - \dfrac{\pi}{6}$. $\quad [\because\ x < 0, y > 0$, point is in 2nd quadrant$]$

Thus, $(r, \theta) = \left(2, \dfrac{5\pi}{6}\right)$ is required polar form.

(4) $(\sqrt{3}, -1)$ **SPU, Sept. 2014, Nov. 2012**

Sol^n. :

Here $x = \sqrt{3}$, $y = -1$

$r = \sqrt{x^2 + y^2}$

$= \sqrt{3+1} = \sqrt{4} = 2$

$\Rightarrow r = 2$

Also $\tan\theta = \dfrac{y}{x} = \dfrac{-1}{\sqrt{3}}$

$\therefore \quad \theta = \tan^{-1}\left(-\dfrac{2}{\sqrt{3}}\right)$

$\Rightarrow \quad \theta = \pi - \dfrac{\pi}{6}$ or $-\dfrac{\pi}{6}$

$\Rightarrow \quad \theta = \dfrac{-\pi}{6}$ $\quad [\because\ x > 0, y < 0$, point is in 4th quadrant$]$

Thus $(r, \theta) = \left(2, -\dfrac{\pi}{6}\right)$ is required polar form.

(5) $(-5, 5)$ **SPU, June-2012**

Sol^n. :

Here $x = -5$, $y = 5$

$r = \sqrt{x^2 + y^2}$

$= \sqrt{25 + 25} = \sqrt{50}$

$\boxed{r = 5\sqrt{2}}$

Also $\tan\theta = \dfrac{y}{x} = \dfrac{5}{-5} = -1$

$\Rightarrow \quad \theta = -\dfrac{\pi}{4}$ or $\pi - \dfrac{\pi}{4}$

$\Rightarrow \quad \boxed{\theta = \pi - \dfrac{\pi}{4}}$ $\quad [\because\ x < 0, y > 0$, 2nd quad.$]$

Thus $(r, \theta) = \left(5\sqrt{2}, \dfrac{3\pi}{4}\right)$ is required polar form.

(6) (0, – 3)

Sol^n. :

$x = 0,\ y = -3$

$\therefore\ r = \sqrt{x^2+y^2} = \sqrt{9}$

$\boxed{r=3}$

Also $\tan\theta = \dfrac{y}{x} = \dfrac{-3}{0} = -\infty$

$\Rightarrow\ \theta = \dfrac{\pi}{2}$ or $\pi + \dfrac{\pi}{2}$

$\therefore\ \theta = \dfrac{3\pi}{2}$ $\quad [\because x = 0,\ y < 0]$

$\therefore\ (r, \theta) = \left(3, \dfrac{3\pi}{2}\right)$ is required polar form.

Ex. 3. Express the following point in Cartesian form (that is in rectangular form).

(1) (3, – 45°)

Sol^n. :

Here, $r = 3,\ \theta = -45°$

$\therefore\ x = r\cos\theta = 3\cos(-45°) = 3\cos 45° = \dfrac{3}{\sqrt{2}}$

and $y = r\sin\theta = 3\sin(-45°) = \dfrac{-3}{\sqrt{2}}$

$\therefore\ (x, y) = \left(\dfrac{3}{\sqrt{2}}, -\dfrac{3}{\sqrt{2}}\right)$ is Cartesian co-ordinates.

(2) (2, 120°) SPU, June-2011

Sol^n. :

Here $r = 2,\ \theta = 120°$

$x = r\cos\theta = r\cos(120°)$

$= 2\cos\cdot(\pi - 60°)$

$= -2\cos 60°$

$= -2\left(\dfrac{1}{2}\right) = -1$

and $y = r\sin\theta = 2\sin(120°) = 2\sin(\pi - 60°)$

$= 2\sin(60°)$

$= 2\cdot\dfrac{\sqrt{3}}{2}$

$= \sqrt{3}$

$\therefore\ (x, y) = (-1, \sqrt{3})$ is Cartesian co-ordinates.

(3) (– 4, 300°) SPU, December-2014

Sol^n. :

Here $r = -4,\ \theta = 300°$

$\therefore\ x = r\cos\theta = -4\cos(300°)$

$= -4\cos(2\pi - 60°)$

$= -4\cos 60°$

$= -4\left(\dfrac{1}{2}\right)$

$= -2$

Also $y = r\sin\theta = -4\sin(2\pi - 60°)$

$= -4\sin 60°$

$= 4\dfrac{\sqrt{3}}{2}$

$= 2\sqrt{3}$

$(x, y) = (-2, 2\sqrt{3})$ is Cartesian co-ordinates.

Ex. 4. Transfer the following equation in Cartesian form.

(1) $r = \dfrac{2}{3\cos\theta + 4\sin\theta}$

Sol^n. :

$r = \dfrac{2}{3\cos\theta + 4\sin\theta}$

$\therefore\ 3r\cos\theta + 4r\sin\theta = 2$

$\therefore\ 3x + 4y = 2$

Which is equation of line.

(2) $r = \tan\theta \cdot \sec\theta$ SPU, June-2012

Sol^n. :

$r = \tan\theta \cdot \sec\theta$

$= \dfrac{y}{x}\dfrac{1}{\cos\theta}$

$\therefore \quad r\cos\theta = \dfrac{y}{x}$

$\therefore \quad x = \dfrac{y}{x}$

$\therefore \quad \boxed{x^2 = y}$ which is equation of parabola.

(3) $r = \tan\theta + \sec\theta$ **SPU, Nov. 2013, Dec. 2014**

Sol^n. :

$r = \tan\theta + \sec\theta$

$\therefore \quad r = \dfrac{y}{x} + \dfrac{1}{\cos\theta}$

$\therefore \quad r = \dfrac{y\cos\theta + x}{x\cos\theta}$

$\therefore \quad xr\cos\theta = y\cos\theta + x$

$\therefore \quad x^2 - x = y\cos\theta$

$\therefore \quad (x^2 - x)\,r = yr\cos\theta$

$\therefore \quad (x^2 - x)\,r = yx$

$\therefore \quad (x-1)\sqrt{x^2+y^2} = y$

$\therefore \quad \sqrt{x^2+y^2} = \dfrac{y}{(x-1)}$

$\therefore \quad \boxed{x^2 + y^2 = \dfrac{y^2}{(x-1)^2}}$

OR

$r = \tan\theta + \sec\theta$

$= \dfrac{\sin\theta}{\cos\theta} + \dfrac{1}{\cos\theta}$

$= \dfrac{\sin\theta + 1}{\cos\theta}$

$\Rightarrow \quad r\cos\theta = \sin\theta + 1$

$\Rightarrow \quad x - 1 = \sin\theta$

$\Rightarrow \quad (x-1)\,r = r\sin\theta$

$\Rightarrow \quad (x-1)^2 r^2 = y^2$

$\Rightarrow \quad \boxed{(x-1)^2(x^2+y^2) = y^2}$

(4) $r = 2\cos\theta + 3\sin\theta$

Sol^n. :

$\therefore \quad r^2 = 2r\cos\theta + 3r\sin\theta$

$\therefore \quad x^2 + y^2 = 2x + 3y$

Which is equation of circle.

(5) $r\,(2\cos\theta + 3\sin\theta) = 4$

Sol^n. :

$\therefore \quad 2r\cos\theta + 3r\sin\theta = 4$

$\therefore \quad 2x + 3y = 4$

(6) $r = \sec\theta + \operatorname{cosec}\theta$ **SPU, April-2015**

$\therefore \quad r = \dfrac{1}{\cos\theta} + \dfrac{1}{\sin\theta}$

$\therefore \quad r = \dfrac{\sin\theta + \cos\theta}{\cos\theta \cdot \sin\theta}$

$\therefore \quad r\cos\theta \cdot \sin\theta = \sin\theta + \cos\theta$

$\therefore \quad r\cos\theta \cdot r\sin\theta = r\sin\theta + r\cos\theta$

$\therefore \quad \boxed{xy = y + x}$

(7) $r = \dfrac{6}{1 + 2\cos\theta}$ **SPU, Dec. 2012, June-2011**

Sol^n. :

$\therefore \quad r + 2r\cos\theta = 6$

$\therefore \quad r + 2x = 6$

$\therefore \quad \sqrt{x^2+y^2} + 2x = 6$

$\therefore \quad \boxed{(x^2+y^2) = (6-2x)^2}$

(8) $\theta = 30°$

Sol^n. :

$\tan\theta = \dfrac{y}{x}$

$\therefore \quad \tan 30° = \dfrac{y}{x}$

$\dfrac{1}{\sqrt{3}} = \dfrac{y}{x}$

$\therefore \quad \boxed{x = \sqrt{3}y}$

Which is equation of line.

(9) $r^2 = \sec^2\theta$ **SPU, June-2012, Nov. 2010**

Sol^n. :

$r^2 = \dfrac{1}{\cos^2\theta}$

$\Rightarrow \quad r^2\cos^2\theta = 1$

$\Rightarrow \quad \boxed{x^2 = 1}$

GRAPH OF SOME POLAR CURVE

To sketch the graph of polar curve we have to discuss following point :

(1) Symmetry

(2) Closeness

(3) Extent

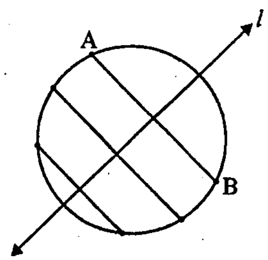
(4) Table of some point.

■ **Closeness :** If we replace θ by $2\pi + \theta$ in $r = f(\theta)$, equation remains unchanged then we say that given curve is close.

■ **Definition :**

(1) Symmetry curve about line l :

A curve is said to be symmetry with respect to line l, if whenever point A lies on the curve then point B which is symmetry to A with respect to line l also lies on the curve.



(2) Symmetry curve about point 0 :

A curve is said to be symmetry with respect to point 0, if whenever point A lies on the curve, the point B which is symmetry to A with respect to point 0 also lies on the curve.

Theorem-1 :

Prove that a curve given by polar equation is symmetry with respect to polar axis, if one of the following condition hold. **SPU, June-2011**

(i) The equation remains unchanged on replacing θ by $-\theta$.

(ii) The equation remains unchanged on replacing r by $-r$ and θ by $\pi - \theta$.

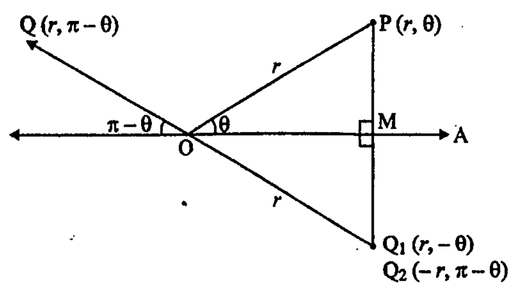
OR

State when a polar curve is symmetry with respect to polar axis ? Also prove it.

Proof : **SPU, April-2016, Nov. 2015, Nov. 2013, Dec. 2012, Nov. 2010**

Let $f(r, \theta) = 0$ be the polar curve.

(i) If equation remains unchanged on replacing θ by $-\theta$, then we have $f(r, -\theta) = 0$. It means that if point $P(r, \theta)$ lies on the curve then point $Q_1(r, -\theta)$ also lies on the curve. .... (1)



From fig. we say that PQ_1 cut the polar axis at M also ΔPOQ_1 is isosceles triangle, with $OP = OQ_1$ and OM bisect $\angle POQ_1$.

$\therefore$ OM is perpendicular bisector of $\overline{PQ_1}$.

$\therefore$ By definition we say that P and Q_1 are symmetry with respect to polar axis. ... (2)

$\therefore$ By (1) and (2) we say that given curve is symmetry about polar axis.

(ii) If equation $f(r, \theta) = 0$ remains unchanged on replacing r by $-r$ and θ by $\pi - \theta$, then we have $f(-r, \pi - \theta) = 0$. It means that if point $P(r, \theta)$ lies on the curve then point $Q_2(-r, \pi - \theta)$ also lies on the curve. .... (3)

Also from the fig. we say that Q_1 and Q_2 are the same point.

$\therefore$ By above case (i), we say that P and Q_2 are symmetry about polar axis. ... (4)

From (3) and (4) we say that given curve is symmetry about polar axis.

Hence, theorem is prove.

Theorem-2 :

Prove that a curve given by polar equation is symmetry with respect to normal axis, if one of the following condition hold. **SPU, December-2012**

(i) The equation remains unchanged on replacing θ by $\pi - \theta$.

(ii) The equation remains unchanged on replacing r by $-r$ and θ by $-\theta$.

OR

State when a polar curve is symmetry with respect to normal axis ? Also prove it.

SPU, April-2016, Nov. 2015, Dec. 2014; April-2015; Nov. 2012, 2013; June-2012

Proof :

Let $f(r, \theta) = 0$ be the polar curve.

(i) If equation remains unchanged on replacing θ by $\pi - \theta$, then we have $f(r, \pi - \theta) = 0$. It means that if point $P(r, \theta)$ lies on the curve then point $Q_1(r, \pi - \theta)$ also lies on the curve. .... (1)

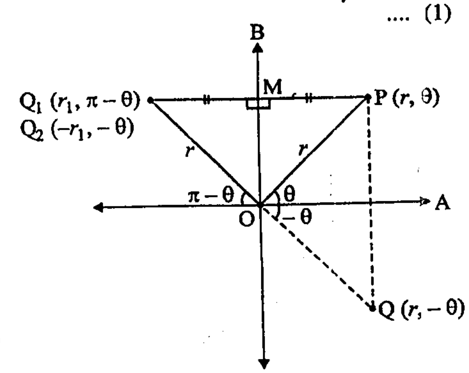
$\rightarrow$ Let PQ_1 cut the normal axis in M as shown in the fig.

Also from the fig. we say that ΔPOQ_1 is isosceles triangle with $OP = OQ_1$ and OM bisect $\angle POQ_1$.

$\therefore$ OM is $\perp^{er}$ bisector of PQ_1.

$\therefore$ Point P and Q_1 are symmetry about normal axis. ... (2)

$\therefore$ From (1) and (2), we say that given curve is symmetry with respect to normal axis.



(ii) If equation remains unchanged on replacing r by $-r$ and θ any $-\theta$, then we have $f(-r, -\theta) = 0$. It means that if P (r, θ) lies on the curve then Q_2 $(-r, -\theta)$ also lies on the curve. .... (3)

From fig. we say that Q_1 and Q_2 are the same point. Therefore by above case (i), we say that P and Q_2 are symmetry point about normal axis. ... (4)

∴ From (3) and (4) we say that given curve is symmetry with respect to normal axis.

Hence theorem is proved.

Theorem-3 : (Only statement)

→ A curve given by polar equation is symmetry with respect to pole, if any one of the following condition is hold.

(i) The equation remains unchanged on replacing r by $-r$.

(ii) The equation remains unchanged on replacing θ by $\pi + \theta$.

OR

State when a polar curve is symmetry with respect to pole ?

■ **Definition :**

(1) Limacon : The curve given by equation $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$ $(a > 0, b > 0)$ are called limacon.

→ If $a = b$, then limacon is called cardiod.

→ If $a > b$, then limacon is surround the pole.

→ If $a < b$, then limacon has inner loop.

(2) Lemniscate : The curve given by $r^2 = \pm a^2\cos 2\theta$ or $r^2 = \pm a^2 \sin 2\theta$ are called lemniscate.

Remark :

(i) Shape of lemniscate is 8 (Eight).

(ii) Lemniscate of the form $r^2 = \pm a^2\cos 2\theta$ is symmetry about polar axis, normal axis and pole.

(iii) Lemniscate of the form $r^2 = \pm a^2 \sin 2\theta$ is symmetry about pole only.

(iv) Limacon of the form $r = a \pm b\cos\theta$ is symmetry about polar axis only.

(v) Limacon of the form $r = a \pm b\sin\theta$ is symmetry about normal axis only.

(3) Rose curve : The curve given by equation $r = a\cos n\theta$ or $r = a\sin n\theta$ $(a > 0, n \in N)$ is called rose curve.

→ If n is odd, then graph contains n loops.

→ If n is even, then graph contains $2n$ loops.

Remark :

(i) In rose curve if n is even then it is symmetry about polar axis, normal axis and pole.

(ii) In $r = a\cos n\theta$, if n is odd then it is symmetry about polar axis.

(iii) In $r = a\sin n\theta$, if n is odd then it is symmetry about normal axis.

(4) **Spiral :** There are three types of spirals.

(i) **Logarithmic spiral :** It is given by $r = e^{a\theta}$,

(ii) **Archimedes spiral :** It is given by $r = a\theta$.

(iii) **Reciprocal spiral :** It is given by $r\theta = a$ i.e. $r = \frac{a}{\theta}$ $(\theta \neq 0)$.

Ex. 5. Sketch the following curve. SPU, April-2016, April-2015, June-2011

(1) $r = 3(1 + \cos\theta)$

Here, $a = 3$, $b = 3$ ∴ $a = b$

Given curve is cardiod.

(i) **Symmetry :** If we replace θ by $-\theta$ equation remains unchanged. So the curve is be symmetry about polar axis, Clearly it is not symmetry about normal axis and pole.

(ii) **Closeness :** If we replace θ by $2\pi + \theta$ equation remains unchanged so given curve is closed. So we can take θ between 0 to 2π.

(iii) **Extent :** We know that,

$-1 \le \cos\theta \le 1$

⇒ $0 \le 1 + \cos\theta \le 2$

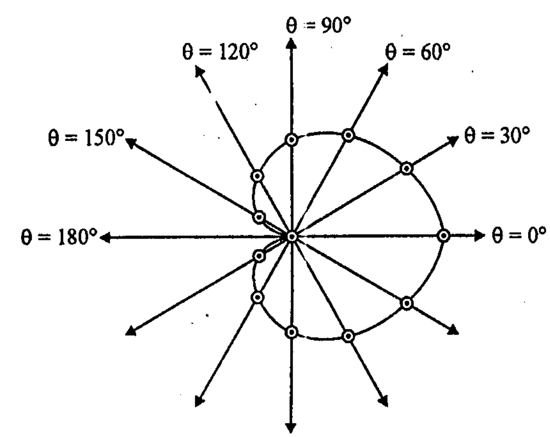
⇒ $0 \le 3(1 + \cos\theta) \le 6$

⇒ $0 \le r \le 6$

We know that given curve is symmetry about polar axis. So we can take θ between 0 to π.

(iv) **Table of some points :**

θ	0	30°	60°	90°	120°	150°	180°
$r = 3(1+\cos\theta)$	6	5.6	4.5	3	1.5	0.4	0



(2) $r = 2 + \cos\theta$

Here, $a = 2$, $b = 1$ $\quad \therefore \; a > b$

Given curve is limacon surround the pole.

(i) **Symmetry :** If we replace θ by $-\theta$, equation remains unchanged so given curve is symmetry about polar axis. Clearly it is not symmetry about normal axis and pole.

(ii) **Closeness :** If we replace θ by $2\pi + \theta$ equation remains unchanged. So given curve is closed. So we can take θ between 0 to 2π.

(iii) **Extent :** We know that

$$-1 \le \cos\theta \le 1$$

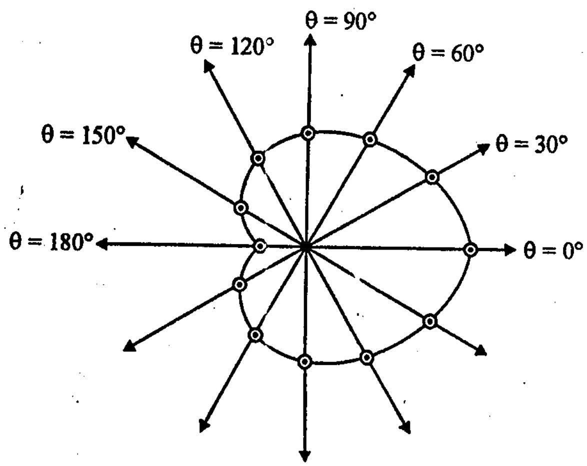
$$\Rightarrow \; 1 \le 2 + \cos\theta \le 3$$

$$\Rightarrow \; 1 \le r \le 3$$

We know that given curve is symmetry about polar axis so we can take θ between 0 to π.

(iv) **Table of some points :**

θ	0	30°	60°	90°	120°	150°	180°
$r = 2 + \cos\theta$	3	2.8	2.5	2	1.5	1.1	1



(3) $r = 2 + 3\cos\theta$ **SPU, June-2012, Dec. 2014**

Here, $a = 2$, $b = 3$ $\quad \therefore \; a < b$

Given curve is limacon with inner loop.

(i) **Symmetry :** If we replace θ by $-\theta$, equation remains unchanged. So the curve is symmetry about polar axis. Clearly it is not symmetry about normal axis and pole.

(ii) **Closeness :** If we replace θ by $2\pi + \theta$ equation remains unchanged. So the curve is closed. So we can take value of θ between 0 to 2π.

(iii) **Extent :** We know that

$$-1 \le \cos\theta \le 1$$

$$\Rightarrow \; -3 \le 3\cos\theta \le 3$$

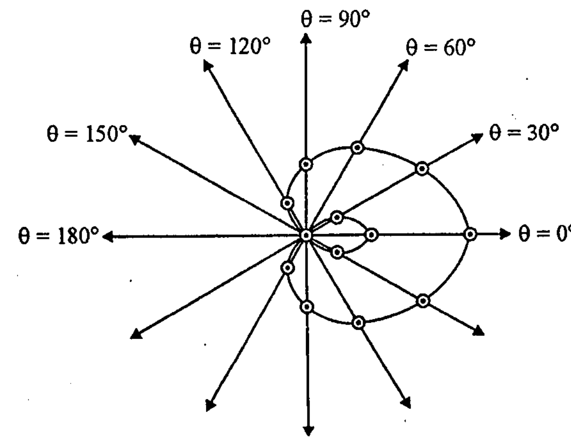
$$\Rightarrow \; -1 \le 2 + 3\cos\theta \le 5$$

$$\Rightarrow \; -1 \le r \le 5$$

The curve is symmetry about polar axis so we can take value of θ between 0 to π.

(iv) **Table of some points :**

θ	0°	30°	60°	90°	120°	150°	180°
$r = 2 + 3\cos\theta$	5	4.6	3.5	2	0.5	−0.6	−1



(4) $r = 3\,(1 - \cos\theta)$

Here, $a = 3$, $b = 3$ $\quad \therefore \; a = b$

Given curve is cardiod.

(i) **Symmetry :** If we replaced θ by $-\theta$, equation remains unchanged. So the given curve is symmetry about polar axis. Clearly it is not symmetry about normal axis and pole.

(ii) **Closeness :** If we replace θ by $2\pi + \theta$ equation remains unchanged. So the curve is closed. So we can take θ between 0 to 2π.

(iii) Extent : We know that

$-1 \leq \cos\theta \leq 1$

$\Rightarrow \quad -3 \leq -\cos\theta \leq 1$

$\Rightarrow \quad 0 \leq 1 - \cos\theta \leq 2$

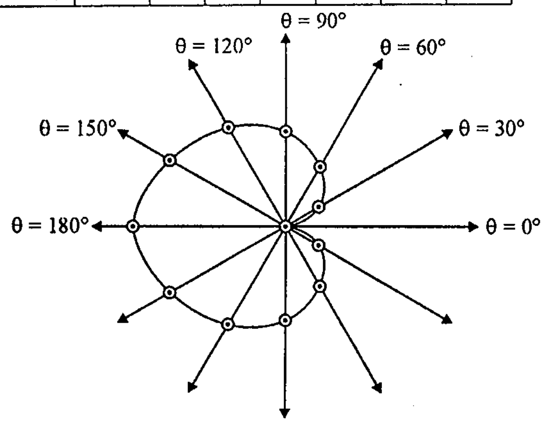
$\Rightarrow \quad 0 \leq 3(1 - \cos\theta) \leq 6$

$\Rightarrow \quad 0 \leq r \leq 6$

The curve is symmetry about polar axis so we can take θ between 0 to π.

(iv) Table of some points :

θ	0°	30°	60°	90°	120°	150°	180°
$r = 3(1 - \cos\theta)$	0	0.4	1.5	3	4.5	5.5	6



(5) $r = 2 - \cos\theta$

Here, $a = 2$, $b = 1 \quad \therefore \quad a > b$

Given curve is limacon surround the pole.

(i) **Symmetry :** If we replaced θ by $-\theta$, equation remains unchanged. So the curve is symmetry about polar axis. Clearly it is not symmetry about normal axis and pole.

(ii) **Closeness :** If we replace θ by $2\pi + \theta$ equation remains unchanged. So the curve is closed. So we can take value of θ between 0 to 2π.

(iii) **Extent :** We know that

$-1 \leq \cos\theta \leq 1$

$\Rightarrow \quad 1 \geq -\cos\theta \geq -1$

$\Rightarrow \quad 3 \geq 2 - \cos\theta \geq 1$

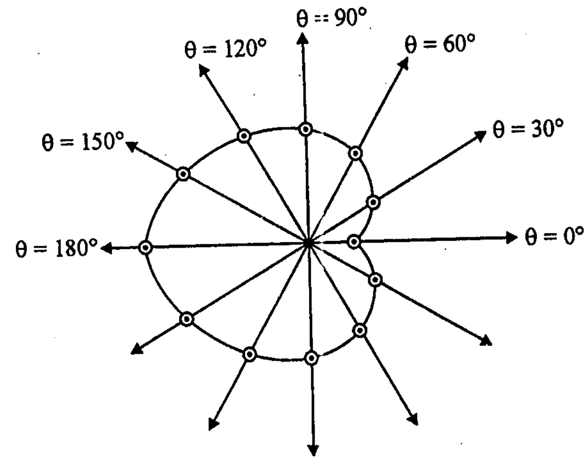
$\Rightarrow \quad 1 \leq 2 - \cos\theta \leq 3$

$\Rightarrow \quad 1 \leq r \leq 3$

We know that given curve is symmetry about polar axis. We can take value of θ between 0 to π.

(iv) Table of some points :

θ	0°	30°	60°	90°	120°	150°	180°
$r = 2 - \cos\theta$	1	1.1	1.5	2	2.5	2.8	3



(6) $r = 2 - 3\cos\theta$

Here, $a = 2$, $b = 3 \quad \therefore \quad a < b$

Given curve is limacon with inner loop.

(i) **Symmetry :** If we replaced θ by $-\theta$, equation remains unchanged. So the curve is symmetry about polar axis. Clearly it is not symmetry about normal axis and pole.

(ii) **Closeness :** If we replace θ by $2\pi + \theta$ equation remains unchanged. So the curve is closed. So we can take θ between 0 to 2π.

(iii) **Extent :** We know that,

$-1 \leq \cos\theta \leq 1$

$\Rightarrow \quad 3 \geq -3\cos\theta \geq -3$

$\Rightarrow \quad 5 \geq 2 - 3\cos\theta \geq -1$

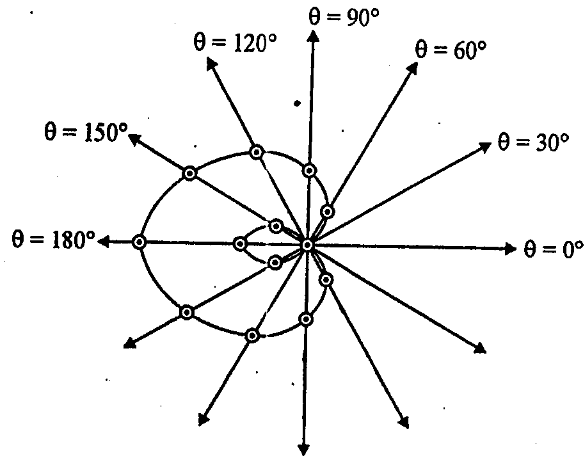
$\Rightarrow \quad -1 \leq 2 - 3\cos\theta \leq 5$

$\Rightarrow \quad -1 \leq r \leq 5$

We know that the curve is symmetry about polar axis. So we can take θ between 0 to π.

(iv) Table of some points :

θ	0°	30°	60°	90°	120°	150°	180°
$r = 2 - 3\cos\theta$	−1	−0.5	0.5	2	3.5	4.5	5



(7) $r^2 = 9\sin 2\theta$ **SPU, Nov. 2016, April-2016 June-2012**

Given curve is lemniscate.

(i) **Symmetry :** If we replaced r by $-r$, equation remains unchanged. So curve is symmetry about pole. Clearly it isn't symmetry about polar axis and normal axis.

(ii) **Closeness :** If we replace θ by $2\pi + \theta$ equation remains unchanged. So the curve is closed. We can take θ between 0 to 2π.

(iii) **Extent :** We know that,

$$-1 \le \sin 2\theta \le 1$$

$$\Rightarrow \quad -9 \le 9\sin 2\theta \le 9$$

$$\Rightarrow \quad -9 \le r^2 \le 9$$

$$\Rightarrow \quad \boxed{-3 \le r \le 3}$$

Also We know that

$$r^2 \ge 0$$

$$\Rightarrow \quad 9\sin 2\theta \ge 0$$

$$\Rightarrow \quad \sin 2\theta \ge 0$$

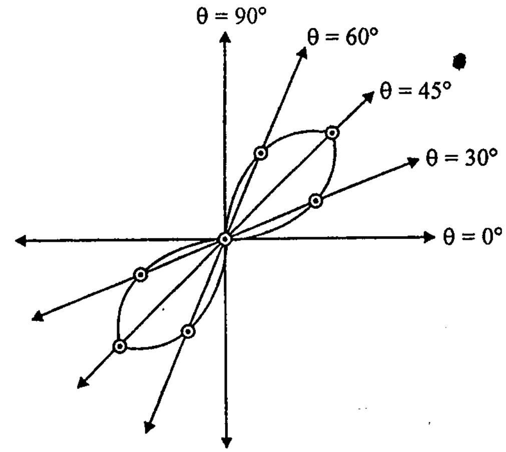
$$\therefore \quad 0 \le 2\theta \le \pi \quad \text{or} \quad 2\pi \le 2\theta \le 3\pi$$

$$\boxed{0 \le \theta \le \frac{\pi}{2}} \quad \text{or} \quad \boxed{\pi \le \theta \le \frac{3\pi}{2}}$$

Given curve is symmetry about pole. So we can take θ between 0 to $\frac{\pi}{2}$.

(iv) **Table of some points :**

θ	0°	30°	45°	60°	90°
$r^2 = 9\sin 2\theta$	0	7.8	9	7.7	0
$r = \pm 3\sqrt{\sin 2\theta}$	0	±2.7	±3	±2.7	0



(8) $r^2 = -16\sin 2\theta$ **SPU, April-2015, June-2011**

Given curve is lemniscate.

(i) **Symmetry :** If we replaced r by $-r$, equation remains unchanged. So the curve is symmetry about pole. Clearly it isn't symmetry about polar axis and normal axis.

(ii) **Closeness :** If we replace θ by $2\pi + \theta$ equation remains unchanged. So given curve is closed.

(iii) **Extent :** We know that,

$$-1 \le \sin 2\theta \le 1$$

$$\Rightarrow \quad 16 \ge -16\sin 2\theta \ge -16$$

$$\Rightarrow \quad -16 \le r^2 \le 16$$

$$\Rightarrow \quad \boxed{-4 \le r \le 4}$$

Also we know that

$$r^2 \ge 0$$

$$\Rightarrow \quad -16\sin 2\theta \ge 0$$

$$\Rightarrow \quad 16\sin 2\theta \le 0$$

$$\Rightarrow \quad \sin 2\theta \le 0$$

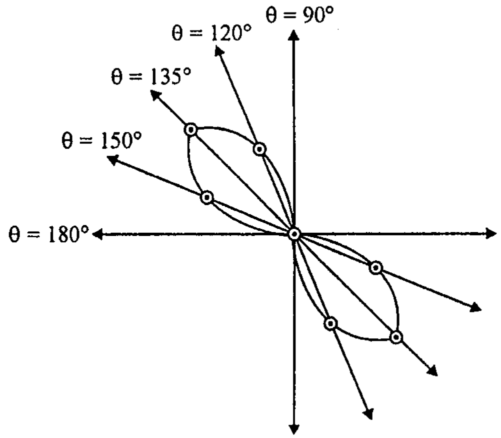
$\Rightarrow \quad \pi \le 2\theta \le 2\pi$ or $3\pi \le 2\theta \le 4\pi$

$\Rightarrow \quad \boxed{\frac{\pi}{2} \le \theta \le \pi}$ or $\boxed{\frac{3\pi}{2} \le \theta \le 2\pi}$

Given curve is symmetry about pole. So we can take θ between $\frac{\pi}{2}$ to π.

(iv) Table :

θ	90°	120°	135°	150°	180°
$r^2 = -16\sin 2\theta$	0	13.85	16	13.85	0
$r = \pm 4\sqrt{-\sin 2\theta}$	0	± 3.72	± 4	± 3.72	0



(9) $r = \cos 2\theta$ **SPU, November-2010**

Given curve is rose curve with 4 loop. ($\because$ $n = 2$, even number)

(i) **Symmetry :** If we replaced θ by $-\theta$, equation remains unchanged. So curve is symmetry about polar axis.

If we replace θ by $\pi - \theta$, equation remains unchanged. So curve is symmetry normal axis.

If we replace θ by $\pi + \theta$, equation remains unchanged. So curve is symmetry about pole.

(ii) **Closeness :** If we replace θ by $2\pi + \theta$ equation remains unchanged. So given curve is closed. We can take θ between 0 to 2π.

(iii) **Extent :** We know that,

$-1 \le \cos 2\theta \le 1$

$\Rightarrow \quad \boxed{-1 \le r \le 1}$

We know that

$\cos 2\theta = 0$

$\therefore \quad 2\theta = \frac{\pi}{2}$

$\therefore \quad \boxed{\theta = \frac{\pi}{4}}$

$\cos 2\theta = 1$

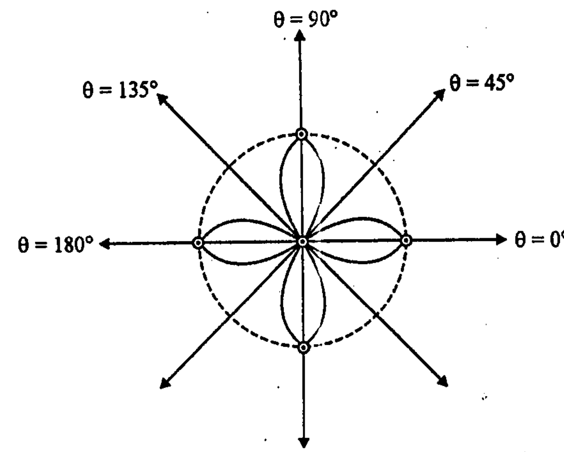
$\therefore \quad 2\theta = 2\pi$

$\therefore \quad \boxed{\theta = \pi}$

Given curve is symmetry about polar axis. We can take θ between 0 to π and difference between θ is $\frac{\pi}{4}$ in table.

(iv) Table :

θ	0°	45°	90°	135°	180°
$r = \cos 2\theta$	1	0	− 1	0	1



(10) $r = \sin 3\theta$ **SPU, November-2013**

Given curve is rose curve with 3 loop. ($\because$ $n = 3$, odd no.)

(i) **Symmetry :** If we replaced θ by $\pi - \theta$, equation remains unchanged. So curve is symmetry about normal axis. Clearly curve is not symmetry about polar axis and pole.

(ii) **Closeness :** If we replace θ by $2\pi + \theta$ equation remains unchanged. So the curve is closed,

(iii) **Extent :** We now that

$-1 \le \sin 3\theta \le 1$

$\therefore \quad -1 \le r \le 1$

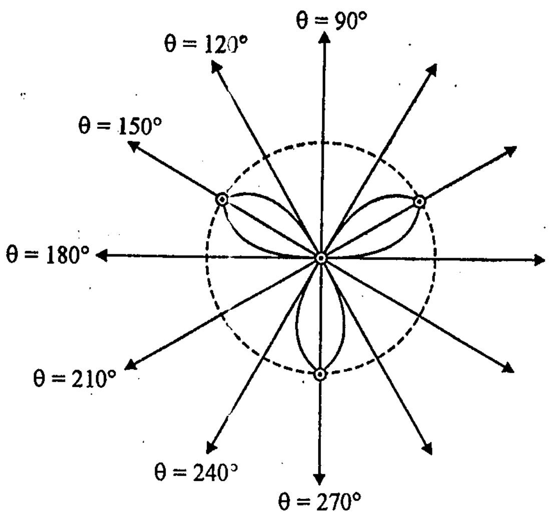
We know that

$\sin 3\theta = 0$	$\sin 3\theta = 1$
$\therefore \quad 3\theta = 0$ or 2π	$\therefore \quad 3\theta = \frac{\pi}{2}$
$\therefore \quad \theta = 0$ or $\frac{\pi}{3}$	$\therefore \quad \theta = \frac{\pi}{6}$

Given curve is symmetry about normal axis. We can take θ between $\frac{\pi}{2}$ to $\frac{3\pi}{2}$ and difference between θ is 30° in table.

(iv) Table :

θ	90°	120°	150°	180°	210°	240°	270°
$r = \sin 3\theta$	−1	0	1	0	−1	0	1



MULTIPLE CHOICE QUESTIONS

■ **Fill in the Blanks :**

1. **For the curve $r^2 = 9\sin 2\theta$ extent is ______.**
   (a) $-3 \le r \le 3$ (b) $-9 \le r \le 9$
   (c) $0 \le r \le 3$ (d) $-2 \le r \le 2$

2. **The curve of $r^2 = 9\sin 2\theta$ is symmetric about ______.**
   (a) Polar axis (b) Normal axis
   (c) Pole (d) Polar axis and normal axis and pole
3. **The curve of $r = \sin 3\theta$ is symmetric about ______.**
   (a) Polar axis (b) Normal axis
   (c) Pole (d) Polar axis and normal axis and pole
4. **The curve of $r = \cos 3\theta$ is symmetric about ______.**
   (a) Polar axis (b) Normal axis
   (c) Pole (d) Polar axis and normal axis and pole
5. **The curve of $r = \cos 2\theta$ is symmetric about ______.**
   (a) Polar axis (b) Normal axis
   (c) Pole (d) Polar axis, normal axis and pole
6. **The curve of $r = \sin 4\theta$ is symmetric about ______.**
   (a) Polar axis (b) Normal axis
   (c) Pole (d) Polar axis, normal axis and pole
7. **The curve of $r = 2\theta$ is symmetric about ______.**
   (a) Polar axis (b) Normal axis
   (c) Pole (d) Polar axis, normal axis and pole
8. **Lemniscate $r^2 = a^2\cos 2\theta$ is symmetric about ______.**
   (a) Polar axis (b) Normal axis
   (c) Pole (d) Polar axis, normal axis and pole
9. **Lemniscate $r^2 = a^2\sin 2\theta$ is symmetric about ______.**
   (a) Polar axis (b) Normal axis
   (c) Pole (d) Polar axis, normal axis and pole
10. **If n is even then Rose curve $r = a\cos n\theta$ symmetric about ______.**
    (a) Polar axis (b) Normal axis
    (c) Pole (d) Polar axis, normal axis and pole
11. **If n is even then Rose curve $r = a\sin n\theta$ symmetric about ______.**
    (a) Polar axis (b) Normal axis
    (c) Pole (d) Polar axis, normal axis and pole
12. **If n is odd then Rose curve $r = a\cos n\theta$ symmetric about ______.**
    (a) Polar axis (b) Normal axis
    (c) Pole (d) Polar axis, normal axis and pole

13. If n is odd then Rose curve $r = a\sin n\theta$ symmetric about ______.

(a) Polar axis (b) Normal axis
(c) Pole (d) Polar axis, normal axis and pole

14. If n is even then Rose curve $r = a\cos n\theta$ has ______ loops.

(a) 1 (b) 2
(c) n (d) $2n$

15. If n is odd then Rose curve $r = a\cos n\theta$ has ______ loops.

(a) 1 (b) 2
(c) n (d) $2n$

16. The curve of $r = a\theta$ is symmetric about ______.

(a) Polar axis (b) Normal axis
(c) Pole (d) Polar axis, normal axis and pole

17. The curve of $r = \sin 3\theta$ has ______ loops.

(a) 1 (b) 2
(c) 3 (d) 6

18. The curve of $r = 3\cos 4\theta$ has ______ loops.

(a) 1 (b) 8
(c) 2 (d) 4

19. $r = \dfrac{2}{(3\cos\theta + 4\sin\theta)}$ represent *a* ______.

(a) line (b) parabola
(c) ellipse (d) circle

20. $r = \tan\theta \sec\theta$ represent *a* ______.

(a) line (b) parabola
(c) ellipse (d) circle

21. $r = 2 + 2\cos\theta$ is equation of ______.

(a) rose curve (b) lemniscate
(c) ellipse (d) limacon

22. The curve $r = 3 \sec\theta \tan\theta$ is an equation of ______.

(a) Ellipse (b) Hyperbola
(c) Line (d) Parabola

23. The curve $r = 2 + 3\sin\theta$ is symmetric with respect to ______.

(a) Pole (b) Polar axis
(c) Line $y = x$ (d) Normal axis

24. The curve $r = 3\cos\theta$ is symmetric with respect to ______.

(a) Pole (b) Polar axis
(c) Line $y = x$ (d) Normal axis

25. The reciprocal of $r = \dfrac{1}{1-\cos\theta}$ is ______.

(a) Circle (b) Rose curve
(c) Cardioid (d) Lemniscate

26. The curve $r = 4\sin 5\theta$ is an equation of ______.

(a) Limacon (b) Rose curve
(c) Cardioid (d) Lemniscate

27. The curve $\theta = 40°$ is an equation of ______.

(a) Ellipse (b) Hyperbola
(c) Line (d) Parabola

28. For the Hyperbola eccentricity ______.

(a) $e = 1$ (b) $e < 1$
(c) $e > 1$ (d) $e = 0$

ANSWERS

1. (a),	**2.** (c),	**3.** (b),	**4.** (a),	**5.** (d),	**6.** (d),	**7.** (b),	**8.** (d),
9. (c),	**10.** (d),	**11.** (d),	**12.** (a),	**13.** (b),	**14.** (d),	**15.** (c),	**16.** (b),
17. (c),	**18.** (b),	**19.** (a),	**20.** (b),	**21.** (d).	**22.** (d),	**23.** (d),	**24.** (b),
25. (c),	**26.** (b),	**27.** (c),	**28.** (c).				

SHORT QUESTIONS

■ **Answer the following :**

1. Express the following points in three other ways such that $-2\pi \le \theta \le 2\pi$.

(a) $(3, -40°)$ (d) $(-3, 30°)$ (g) $(5, 225°)$
(b) $(2, -65°)$ (e) $(3, 45°)$ (h) $(-4, 135°)$
(c) $(-3, -30°)$ (f) $(2, -\pi)$ (i) $(-5, -200°)$

2. Express the following points in polar form.

(a) $(1, \sqrt{3})$ (c) $(1, -\sqrt{3})$ (e) $(5, -5)$
(b) $(-1, \sqrt{3})$ (d) $(-1, -\sqrt{3})$ (f) $(3, 0)$

3. Express the following points in cartesian form. (i.e. rectangular form)

(a) (3, 45°) (c) (−3, 0°) (e) (4, − 300°)

(b) (2, − 120°) (d) (−3, 125°)

4. Express the following equations in cartesian form.

(a) $r = 2/(3\cos\theta - 4\sin\theta)$

(b) $r = \cot\theta \operatorname{cosec}\theta$

(c) $r = \cot\theta - \operatorname{cosec}\theta$

(d) $r = 2\cos\theta - 3\sin\theta$

(e) $r(2\cos\theta - 3\sin\theta) = 4$

(f) $r = \sec\theta - \operatorname{cosec}\theta$

(g) $r = 6/(1 - 2\cos\theta)$

(h) $r^2 = \sec 2\theta$

(i) $r^2 = 1/(\sin 2\theta - \cos 2\theta)$

(j) $r = 4/(2 + \cos\theta)$

(k) $r\cos\theta = 4$

(l) $r = 4\cos\theta$

(m) $\theta = 120°$

EXERCISE

1. Prove that a curve given by a polar equation is symmetric with respect to pole, if one of the following condition hold;

(1) The equation remains unchanged on replacing r by $-r$.

(2) The equation remains unchanged on replacing θ by $\pi + \theta$.

OR

State when a polar curve is symmetric with respect to pole. Prove it.

2. Give all the symmetry in the following problems and draw the graph.

OR

Sketch the curve given by following.

OR

Discuss symmetry, closeness, extent, table of some points for the following. Hence sketch the curve.

(a) $r = 3(1 - \sin\theta)$

(b) $r = 2 - 3\sin\theta$

(c) $r = 2 - \sin\theta$

(d) $r = 3 + 2\sin\theta$

(e) $r = 2 + 3\sin\theta$

(f) $r^2 = 4\sin 2\theta$

(g) $r^2 = -4\sin 2\theta$

(h) $r^2 = 9\cos 2\theta$

(i) $r^2 = -4\cos 2\theta$

(j) $r = 3\cos 2\theta$

(k) $r = 4\sin 2\theta$

(l) $r = 4\cos 3\theta$ **[SPU, April-2015]**

(m) $r = 2\sin 5\theta$

(n) $r = 2\cos 4\theta$

(o) $r = 3\sin 4\theta$

(p) $r = \cos^2\theta$

(q) $r = 2|\cos\theta|$

POLAR EQUATION OF CONIC

Theorem-1 :

Consider a conic whose one focus is at the pole.

(1) Obtain equation of conic, where the directrix is perpendicular to the polar axis.

(2) Obtain equation of conic, where the directrix is parallel to the polar axis.

OR

In usual notation prove that

(1) $r = \dfrac{pe}{1 \pm e\cos\theta}$ **SPU, April-2015, 2016, Nov. 2013, Dec. 2012, June-2011, Nov. 2010**

(2) $r = \dfrac{pe}{1 \pm e\sin\theta}$ **SPU, Nov. 2015, Dec. 2014, Nov. 2012, June-2012**

Proof :

(1) Case-I :

If directrix is perpendicular to the polar axis and right to the pole at distance p from the pole.

Let focus F be at pole O.

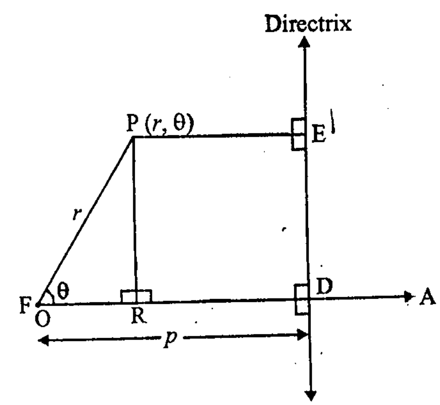
Let P (r, θ) be any point on conic, then draw $\overline{PE} \perp$ directrix, $\overline{PR} \perp$ polar axis, $\overline{OD} \perp$ directrix.

From figure we say that OP = r, ∠DOP = θ, OD = p.

From right angle triangle, Δ POR

$$\cos\theta = \frac{OR}{OP} = \frac{OR}{r} \Rightarrow OR = r\cos\theta$$

and $\sin\theta = \dfrac{PR}{r} \Rightarrow PR = r\sin\theta$



We know that eccentricity,

$$e = \frac{d\,(pt\ P,\ focus\ F)}{d\,(pt\ P,\ directrix)}$$

$$e = \frac{PF}{PE} = \frac{r}{RD} = \frac{r}{OD - OR}$$

$$\therefore\quad e = \frac{r}{p - r\cos\theta}$$

$\therefore\quad pe - er\cos\theta = r$

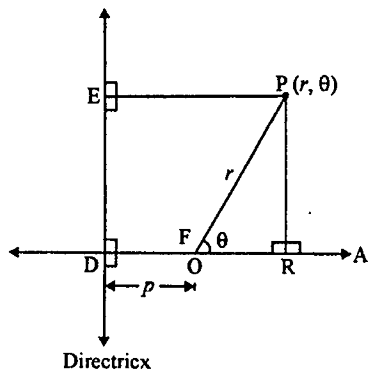
$\therefore\quad pe = r(1 + e\cos\theta)$

$$\therefore\quad \boxed{r = \frac{pe}{1 + e\cos\theta}}$$

(2) Case-II :

If directrix is perpendicular to the polar axis and left to the pole at distance p from the pole, then,

$$e = \frac{PF}{PE} = \frac{r}{RD} = \frac{r}{OD + OR} = \frac{r}{p + r\cos\theta}$$



$\Rightarrow$ $ep + er\cos\theta = r$

$\Rightarrow$ $ep = r(1 - e\cos\theta)$

$$\boxed{r = \frac{ep}{1 - e\cos\theta}}$$

Hence, $\boxed{r = \dfrac{ep}{1 \pm e\cos\theta}}$

Proof-2 :

(1) Case-I :

If directrix is parallel to polar axis and above the pole at distance p from the pole.

Let focus F be at pole O. Let P (r, θ) be any point on conic.

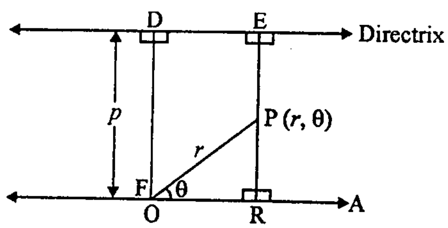
Draw PE $\perp$ directrix

OD $\perp$ directrix

PR $\perp$ polar axis

Here OP $= r$, $\angle$ AOP $= \theta$, OD $= p$

From Δ OPR,



$\cos\theta = \dfrac{OR}{r}$ $\Rightarrow$ OR $= r\cos\theta$

and $\sin\theta = \dfrac{PR}{r}$ $\Rightarrow$ PR $= r\sin\theta$

We know that, eccentricity

$\Rightarrow$ $e = \dfrac{PF}{PE} = \dfrac{r}{ER - PR} = \dfrac{r}{p - r\sin\theta}$ $\quad$ ($\because$ ER $=$ OD $= p$)

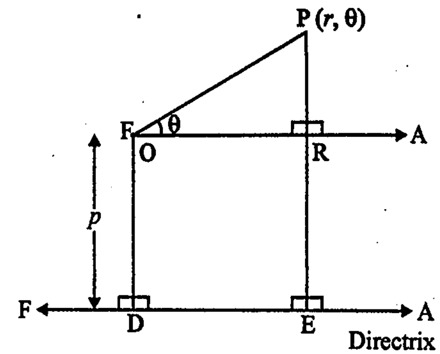
$\Rightarrow$ $ep - er\sin\theta = r$

$\Rightarrow$ $ep = r(1 + e\sin\theta)$

$\Rightarrow$ $\boxed{r = \dfrac{ep}{1 + e\sin\theta}}$

Case-II : If directrix is parallel to polar axis and below the pole at distance p from the pole.

We know that,



$\Rightarrow$ $e = \dfrac{PF}{PE}$

$= \dfrac{r}{PR + RE}$

$= \dfrac{r}{p + r\sin\theta}$

$\Rightarrow$ $ep + er\sin\theta = r$

$\Rightarrow$ $ep = r(1 - e\sin\theta)$

$\therefore$ $\boxed{r = \dfrac{ep}{1 - e\sin\theta}}$

Hence $\boxed{r = \dfrac{ep}{1 \pm e\sin\theta}}$

Remarks :

→ If $e = 1$ then conic is parabola.

→ If $e < 1$ then conic is ellipse.

→ If $e > 1$ then conic is hyperbola.

Ex. 1. Find polar equation of conic if directrix passes through the following points.

(1) (5, 0°) and $e = 1$

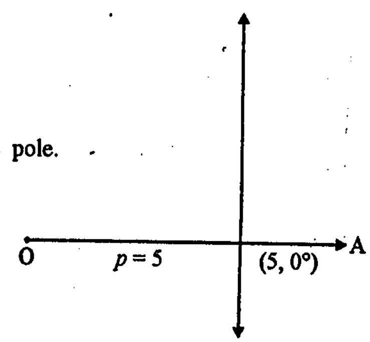
Sol^n. :

Here, $p = 5$, $e = 1$

Here, directrix is $\perp^{er}$ to polar axis and right to the pole.

$\therefore$ Equation of conic is $r = \dfrac{pe}{1 + e\cos\theta}$

$\Rightarrow$ $\boxed{r = \dfrac{5}{1 + \cos\theta}}$



(2) $\left(5, \frac{\pi}{2}\right)$ and $e = \frac{2}{3}$ **SPU, December-2014**

Sol^n^. :

Here, $e = \frac{2}{3}$, $p = 5$

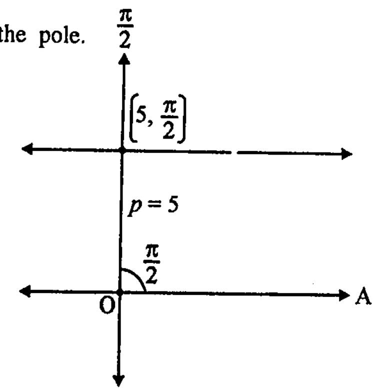
Here directrix is parallel to polar axis and above the pole.

$\therefore$ Equation of conic is

$$r = \frac{pe}{1 + e\sin\theta}$$

$$= \frac{5 \cdot \frac{2}{3}}{1 + \frac{2}{3}\sin\theta}$$

$$\therefore \boxed{r = \frac{10}{3 + 2\sin\theta}}$$



(3) $\left(5, \frac{3\pi}{2}\right)$ and $e = 3$ **SPU, Sept. 2014, June-2011**

Sol^n^. :

Here, $e = 3$, $p = 5$

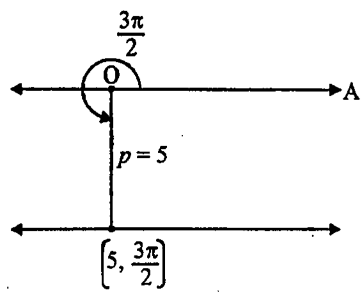
Here directrix is parallel to polar axis and below the pole.

$\therefore$ Equation of conic is

$$r = \frac{pe}{1 - e\sin\theta}$$

$$r = \frac{5 \times 3}{1 - 3\sin\theta}$$

$$\therefore \boxed{r = \frac{15}{1 - 3\sin\theta}}$$



(4) $(-2, \pi)$ and $e = 1.5$

Sol^n^. :

Here, $e = 1.5$, $p = 2$ ($\because$ distance p is positve)

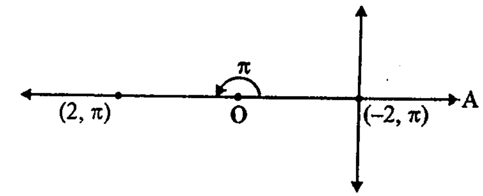
Here directrix is perpendicular to polar axis and right to the pole.

$\therefore$ Equation of conic is

$$r = \frac{pe}{1 + e\cos\theta}$$

$$r = \frac{2\,(1.5)}{1 + 1.5\cos\theta}$$

$$\therefore \boxed{r = \frac{3}{1 + 1.5\cos\theta}}$$



MULTIPLE CHOICE QUESTIONS

■ **Fill in the Blanks :**

1. **If eccentricity $e = 1$ then conic is _____.**
   (a) Hyperbola (b) Parabola
   (c) Circle (d) Ellipse

2. **If eccentricity $e < 1$ then conic is _____.**
   (a) Hyperbola (b) Parabola
   (c) Circle (d) Ellipse

3. **If eccentricity $e > 1$ then conic is _____.**
   (a) Hyperbola (b) Parabola
   (c) Circle (d) Ellipse

4. **If directrix is perpendicular to the polar axis ad left to the pole at distance p from the pole then equation of conic is _____.**
   (a) $r = \frac{pe}{1 + e\cos\theta}$ (b) $r = \frac{pe}{1 + e\sin\theta}$
   (c) $r = \frac{pe}{1 - e\cos\theta}$ (d) $r = \frac{pe}{1 - e\sin\theta}$

5. **If directrix is perpendicular to the polar axis and right to the pole at distance p from the pole then equation of conic is _____.**
   (a) $r = \frac{pe}{1 + e\cos\theta}$ (b) $r = \frac{pe}{1 + e\sin\theta}$
   (c) $r = \frac{pe}{1 - e\cos\theta}$ (d) $r = \frac{pe}{1 - e\sin\theta}$

6. **If directrix is parallel to the polar axis and above the pole at distance p from the pole then equation of conic is ______.**

(a) $r = \dfrac{pe}{1 + e\cos\theta}$ (b) $r = \dfrac{pe}{1 + e\sin\theta}$

(c) $r = \dfrac{pe}{1 - e\cos\theta}$ (d) $r = \dfrac{pe}{1 - e\sin\theta}$

7. **If directrix is parallel to the polar axis and below the pole at distance p from the pole then equation of conic is ______.**

(a) $r = \dfrac{pe}{1 + e\cos\theta}$ (b) $r = \dfrac{pe}{1 + e\sin\theta}$

(c) $r = \dfrac{pe}{1 - e\cos\theta}$ (d) $r = \dfrac{pe}{1 - e\sin\theta}$

8. **Polar equation of conic, if directrix passes through the point $\left(3, \dfrac{\pi}{2}\right)$ and $e = \dfrac{5}{4}$ is ______.**

(a) $r = \dfrac{15}{4 + 5\sin\theta}$ (b) $r = \dfrac{15}{3 - 4\sin\theta}$

(c) $r = \dfrac{15}{4 - 5\sin\theta}$ (d) $r = \dfrac{15}{5 + 4\sin\theta}$

9. **The curve $r = \dfrac{5}{3 + \sin\theta}$ is an equation of ______.**

(a) Ellipse (b) Hyperbola

(c) Line (d) Parabola

10. **The curve $r = \dfrac{2}{1 + \cos\theta}$ is an equation of ______.**

(a) Ellipse (b) Hyperbola

(c) Line (d) Parabola

ANSWERS

1. (b), 2. (d), 3. (a), 4. (c), 5. (a), 6. (b), 7. (d), 8. (a), 9. (a), 10. (b).

SHORT QUESTIONS

4. **Find the polar equation of conic if directrix passes through the point.**

(a) $(5, 0)$ and $e = \dfrac{1}{2}$ (b) $\left(5, \dfrac{\pi}{2}\right)$ and $e = \dfrac{3}{2}$

(c) $\left(5, \dfrac{3\pi}{2}\right)$ and $e = 1$ (d) $(-2, \pi)$ and $e = \dfrac{1}{2}$

(e) $\left(6, \dfrac{3\pi}{2}\right)$ and $e = 1$ (f) $\left(3, -\dfrac{\pi}{2}\right)$ and $e = \dfrac{5}{4}$

EXERCISE

1. **Identify following conic. Find intercepts and symmetry, if any. Also sketch it.**

(a) $r = \dfrac{6}{2 + \cos\theta}$ (e) $r = \dfrac{10}{3 - 2\sin\theta}$ (i) $r = \dfrac{9}{1 + 2\sin\theta}$

(b) $r = \dfrac{6}{2 - \cos\theta}$ (f) $r = \dfrac{6}{1 + 2\cos\theta}$ (j) $r = \dfrac{15}{3 - 7\sin\theta}$

(c) $r = \dfrac{6}{4(1 + \cos\theta)}$ (g) $r = \dfrac{5}{1 - \cos\theta}$ (k) $r = \dfrac{15}{4 + 5\cos\theta}$

(d) $r = \dfrac{4}{1 + \sin\theta}$ (h) $r = \dfrac{3}{1 + 2\sin\theta}$

2. **Identify following curve. Also sketch them.**

(a) $r = \dfrac{1}{1 + \sin\theta}$ (i) $r = \dfrac{12}{2 + \cos\theta}$ (p) $r = 2 + \sin\theta$

(b) $r = \dfrac{1}{2 - 3\cos\theta}$ (j) $r = \dfrac{10}{2 - \sin\theta}$ (q) $r = 1 + \sin\theta$

(c) $r = \dfrac{3}{3 + \cos\theta}$ (k) $r = \dfrac{15}{3 + 7\sin\theta}$ (r) $r = 1 + 2\sin\theta$

(d) $r = \dfrac{8}{4 - 3\sin\theta}$ (l) $r = \dfrac{3}{1 + 2\sin\theta}$ (s) $r = 3 - 2\cos\theta$

(e) $r = \dfrac{4}{2 - 3\cos\theta}$ (k) $r = \dfrac{15}{3 + 7\sin\theta}$ (t) $r = 1 - \sin\theta$

(f) $r = \dfrac{4}{1 + \sin\theta}$ (m) $r = \dfrac{10}{3 - 2\cos\theta}$ (u) $r = 3\cos\theta$

(g) $r = \dfrac{6}{1 - \cos\theta}$ (n) $r = 1 - \cos\theta$ (v) $r = -2\sin\theta$

(h) $r = \dfrac{8}{3 + 3\sin\theta}$ (o) $r = 2 - \cos\theta$

■ ✻ ■

UNIT 3

Reduction Formulae, Volume of a Solid of Revolution, Rectification, Area of a Surface of Revolution

REDUCTION FORMULAE

Reduction formulae are used repeatedly to express the integral of a complicated function in terms of simple ones. The reduction formulae are usually obtained by the rule of integration by parts. We propose to give ready methods for the reproduction of some of the more important reduction formulae. In the sections that follow, we develop the reduction formulae for the integrals involving certain type of trigonometric functions. Most of the formulae are divided into two parts – integral with and without limits. The former will be denoted by J and the latter will be denoted by I, as a convention. While continuously discussing indefinite integrals, for brevity, we may not mention the constants of integration, the presence of which is always assumed. Thus for example, we may write $\int \sin x\, dx = -\cos x$, instead of $\int \sin x\, dx = -\cos x + C$.

INTEGRAL OF $\sin^n x$ and $\cos^n x$

First we obtain reduction formulae for

$$I_n = \int \sin^n x\, dx \text{ and } J_n = \int_0^{\pi/2} \sin^n x\, dx \text{ where } n \in N$$

$$\begin{aligned}
I_n &= \int \sin^n x\, dx \\
&= \int \sin^{n-1} x \sin x\, dx \\
&= \sin^{n-1} x(-\cos x) - \int (n-1)\sin^{n-2} x \cos x(-\cos x)\, dx \\
&= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,(1-\sin^2 x)\, dx \\
&= -\sin^{n-1} x \cos x + (n-1)\, I_{n-2} - (n-1)\, I_n
\end{aligned}$$

$$\Rightarrow \quad nI_n = -\sin^{n-1} x \cos x + (n-1)\, I_{n-2}$$

$$\Rightarrow \quad I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2} \qquad \dots (1)$$

By (1) we get,

$$J_n = -\left[\frac{\sin^{n-1} x \cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n} J_{n-2}$$

$$= \frac{n-1}{n} J_{n-2} \qquad \dots (2)$$

For $n = 3$ and $n = 2$, (2) reduces to

$$J_3 = \frac{2}{3} J_1 = \frac{2}{3}\int_0^{\pi/2} \sin x\, dx = \frac{2}{3}$$

and $$J_2 = \frac{1}{2} J_0 = \frac{1}{2}\int_0^{\pi/2} \sin^0 x\, dx = \frac{1}{2}\frac{\pi}{2}.$$

Now, let n be an odd positive integer. Then by (2), we have,

$$\begin{aligned}
J_n &= \frac{n-1}{n} J_{n-2} \\
&= \frac{(n-1)}{n}\frac{(n-3)}{(n-2)} J_{n-4} \\
&\dots \quad \dots \quad \dots \\
&= \frac{(n-1)}{n}\frac{(n-3)}{(n-2)} \dots J_3 \\
&= \frac{(n-1)}{n}\frac{(n-3)}{(n-2)} \dots \frac{2}{3}
\end{aligned}$$

Now, let n be an even positive integer. Then by (2), we have,

$$\begin{aligned}
J_n &= \frac{n-1}{n} J_{n-2} \\
&= \frac{(n-1)}{n}\frac{(n-3)}{(n-2)} J_{n-4} \\
&\dots \quad \dots \quad \dots \\
&= \frac{(n-1)}{n}\frac{(n-3)}{(n-2)} \dots J_2 \\
&= \frac{(n-1)}{n}\frac{(n-3)}{(n-2)} \dots \frac{1}{2}\frac{\pi}{2}
\end{aligned}$$

Hence, $$J_n = \begin{cases} \dfrac{(n-1)(n-3)\dots 2}{n(n-2)\dots 3} & \text{if } n \text{ is odd;} \\ \dfrac{(n-1)(n-3)\dots 1}{n(n-2)\dots 2}\dfrac{\pi}{2} & \text{if } n \text{ is even.} \end{cases}$$

Now we find $I_n = \int \cos^n x\, dx$ **and** $J_n = \int_0^{\pi/2} \cos^n x\, dx$**, for** $n \in N$.

$$I_n = \int \cos^n x\, dx = \int \cos^{n-1} x \cos x\, dx$$

Integrating by parts and proceeding as earlier we get,

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}$$

and
$$J_n = \begin{cases} \dfrac{(n-1)(n-3)\ldots 2}{n(n-2)\ldots 3} & \text{if } n \text{ is odd;} \\ \dfrac{(n-1)(n-3)\ldots 1}{n(n-2)\ldots 2}\dfrac{\pi}{2} & \text{if } n \text{ is even.} \end{cases}$$

Example-1 : Evaluate $\int \cos^7 x\, dx$

Solution :

$$I_7 = \int \cos^7 x\, dx$$

$$= \frac{\cos^6 x \sin x}{7} + \frac{6}{7} I_5$$

Further, $$I_5 = \frac{\cos^4 x \sin x}{5} + \frac{4}{5} I_3,$$

where, $$I_3 = \frac{\cos^2 x \sin x}{3} + \frac{2}{3} I_1$$

$$= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\, dx$$

$$= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\sin x$$

$$\Rightarrow \quad I_5 = \frac{\cos^4 x \sin x}{5} + \frac{4\cos^2 x \sin x}{15} + \frac{8}{15}\sin x$$

$$\Rightarrow \quad I_7 = \frac{\cos^6 x \sin x}{7} + \frac{6\cos^4 x \sin x}{35} + \frac{8\cos^2 x \sin x}{35} + \frac{16 \sin x}{35}$$

Example-2 : Evaluate $\int_0^{\pi/2} \sin^{10} x\, dx$

Solution :

$$\int_0^{\pi/2} \sin^{10} x\, dx = \frac{9\cdot 7\cdot 5\cdot 3\cdot 1}{10\cdot 8\cdot 6\cdot 4\cdot 2}\frac{\pi}{2} = \frac{63\pi}{512}$$

Example-3 : Evaluate $\int_0^{\pi} \frac{\sin^4\theta}{(1+\cos\theta)^2}\, d\theta$.

Solution :

we can write the given integral as

$$\int_0^{\pi} \frac{\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^4}{\left(2\cos^2\frac{\theta}{2}\right)^2}\, d\theta = \int_0^{\pi} 4\sin^4\frac{\theta}{2}\, d\theta$$

Let $\frac{\theta}{2} = t$. Then $d\theta = 2dt$. Also, $\theta = 0 \Rightarrow t = 0$ and $\theta = \pi \Rightarrow t = \frac{\pi}{2}$.

Hence, the given integral becomes $4\int_0^{\pi/2} \sin^4 t\, 2dt = 8\,\frac{3.1}{4.2}\,\frac{\pi}{2} = \frac{3\pi}{2}$.

Example-4 : Evaluate $\int_0^1 x^5 \sqrt{\frac{1+x^2}{1-x^2}}\, dx$

Solution :

Let $x^2 = \cos\theta$. Then $x\,dx = -\frac{1}{2}\sin\theta\, d\theta$. Also $x = 0 \Rightarrow \theta = \frac{\pi}{2}$ and $x = 1 \Rightarrow \theta = 0$.

Hence the given integral becomes

$$-\frac{1}{2}\int_{\pi/2}^{0} \cos^2\theta \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\sin\theta\, d\theta = \frac{1}{2}\int_0^{\pi/2} \cos^2\theta \sqrt{\frac{2\cos^2\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}}\, 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos^2\theta\,(1+\cos\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos^2\theta\, d\theta + \frac{1}{2}\int_0^{\pi/2} \cos^3\theta\, d\theta$$

$$= \frac{1}{2}\,\frac{1}{2}\,\frac{\pi}{2} + \frac{1}{2}\,\frac{2}{3}$$

$$= \frac{\pi}{8} + \frac{1}{3}$$

$$= \frac{3\pi + 8}{24}$$

Example-5 : Evaluate $\int_0^\infty \frac{x^2}{\sqrt{(1+x^6)^7}}\,dx$

Solution :

Let $x^3 = \tan\theta$, Then $3x^2dx = \sec^2\theta\, d\theta$. Also, $x = 0 \Rightarrow \theta = 0$ and $x \to \infty \Rightarrow \theta \to \frac{\pi}{2}$.

Hence the given integral becomes

$$\frac{1}{3}\int_0^{\pi/2}\frac{\sec^2\theta}{\sec^7\theta}\,d\theta = \frac{1}{3}\int_0^{\pi/2}\cos^5\theta\,d\theta = \frac{1}{3}\frac{4\cdot 2}{5\cdot 3} = \frac{8}{45}.$$

Example-6 : Evaluate $I = \int_0^1 x^5 \sin^{-1} x\, dx$.

Solution :

$$I = \int_0^1 (\sin^{-1} x)\, x^5\, dx$$

$$= \left[\sin^{-1} x\,\frac{x^6}{6}\right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}}\,\frac{x^6}{6}\,dx$$

$$= \frac{\pi}{12} - \frac{1}{6}\int_0^1 \frac{x^6}{\sqrt{1-x^2}}\,dx$$

Let $x = \sin\theta$. Then $dx = \cos\theta\, d\theta$. Also, $x = 0 \Rightarrow \theta = 0$ and $x = 1 \Rightarrow \theta = \frac{\pi}{2}$.

Hence, $$I = \frac{\pi}{12} - \frac{1}{6}\int_0^{\pi/2}\frac{\sin^6\theta}{\cos\theta}\cos\theta\,d\theta$$

$$= \frac{\pi}{12} - \frac{1}{6}\int_0^{\pi/2}\sin^6\theta\,d\theta$$

$$= \frac{\pi}{12} - \frac{1}{6}\,\frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\,\frac{\pi}{2} = \frac{\pi}{12} - \frac{5\pi}{192} = \frac{11\pi}{192}.$$

INTEGRAL OF $\sin^p x \cos^q x$

Now we obtain the reduction formula for $I_{p,q} = \int \sin^p x \cos^q x\, dx$ and $J_{p,q} = \int_0^{\pi/2} \sin^p x \cos^q x\, dx$, where p, q are positive integers.

$$I_{p,q} = \int \sin^p x \cos^q x\, dx$$

$$= \int \sin^{p-1} x\,(\sin x \cos^q x)\, dx$$

$$= \sin^{p-1} x\left(-\frac{\cos^{q+1} x}{q+1}\right) + \int (p-1)\sin^{p-2} x\, \cos x\, \frac{\cos^{q+1} x}{q+1}\,dx$$

$$= -\frac{\sin^{p-1} x \cos^{q+1} x}{q+1} + \frac{p-1}{q+1}\int \sin^{p-2} x \cos^q x\,(1-\sin^2 x)\,dx$$

$$= -\frac{\sin^{p-1} x \cos^{q+1} x}{q+1} + \frac{p-1}{q+1} I_{p-2,q} - \frac{p-1}{q+1} I_{p,q}$$

Hence,

$$\frac{p+q}{q+1} I_{p,q} = -\frac{\sin^{p-1} x \cos^{q+1} x}{q+1} + \frac{p-1}{q+1} I_{p-2,q}$$

$$\Rightarrow \quad I_{p,q} = -\frac{\sin^{p-1} x \cos^{q+1} x}{p+q} + \frac{p-1}{p+q} I_{p-2,q} \qquad \ldots (1)$$

Also, $I_{p,q} = \int \sin^p x \cos^q x\, dx$

$$= \int \cos^{q-1} x\,(\cos x \sin^p x)\,dx$$

Integrating by parts and proceeding as earlier we get,

$$I_{p,q} = \frac{\sin^{p+1} x \cos^{q-1} x}{p+q} + \frac{q-1}{p+q} I_{p,q-2} \qquad \ldots (2)$$

Now we find $J_{p,q} = \int_0^{\pi/2} \sin^p x \cos^q x\, dx$, where p, q are positive integers.

From (1), we get,

$$J_{p,q} = -\left[\frac{\sin^{p-1} x \cos^{q+1} x}{p+q}\right]_0^{\pi/2} + \frac{p-1}{p+q} J_{p-2,q}$$

$$= \frac{p-1}{p+q} J_{p-2,q} \qquad \ldots (3)$$

Example-5 : Evaluate $\int_0^\infty \frac{x^2}{\sqrt{(1+x^6)^7}}\,dx$

Solution :

Let $x^3 = \tan\theta$, Then $3x^2 dx = \sec^2\theta\, d\theta$. Also, $x = 0 \Rightarrow \theta = 0$ and $x \to \infty \Rightarrow \theta \to \frac{\pi}{2}$.

Hence the given integral becomes

$$\frac{1}{3}\int_0^{\pi/2} \frac{\sec^2\theta}{\sec^7\theta}\,d\theta = \frac{1}{3}\int_0^{\pi/2}\cos^5\theta\, d\theta = \frac{1}{3}\frac{4\cdot 2}{5\cdot 3} = \frac{8}{45}.$$

Example-6 : Evaluate $I = \int_0^1 x^5 \sin^{-1} x\, dx$.

Solution :

$$I = \int_0^1 (\sin^{-1} x)\, x^5\, dx$$

$$= \left[\sin^{-1} x\, \frac{x^6}{6}\right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}}\frac{x^6}{6}\,dx$$

$$= \frac{\pi}{12} - \frac{1}{6}\int_0^1 \frac{x^6}{\sqrt{1-x^2}}\,dx$$

Let $x = \sin\theta$. Then $dx = \cos\theta\, d\theta$. Also, $x = 0 \Rightarrow \theta = 0$ and $x = 1 \Rightarrow \theta = \frac{\pi}{2}$.

Hence, $$I = \frac{\pi}{12} - \frac{1}{6}\int_0^{\pi/2}\frac{\sin^6\theta}{\cos\theta}\cos\theta\, d\theta$$

$$= \frac{\pi}{12} - \frac{1}{6}\int_0^{\pi/2}\sin^6\theta\, d\theta$$

$$= \frac{\pi}{12} - \frac{1}{6}\frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\frac{\pi}{2} = \frac{\pi}{12} - \frac{5\pi}{192} = \frac{11\pi}{192}.$$

INTEGRAL OF $\sin^p x \cos^q x$

Now we obtain the reduction formula for $I_{p,q} = \int \sin^p x \cos^q x\, dx$ and $J_{p,q} = \int_0^{\pi/2} \sin^p x \cos^q x\, dx$, where p, q are positive integers.

$$I_{p,q} = \int \sin^p x \cos^q x\, dx$$

$$= \int \sin^{p-1} x\, (\sin x \cos^q x)\, dx$$

$$= \sin^{p-1} x\left(-\frac{\cos^{q+1} x}{q+1}\right) + \int (p-1)\sin^{p-2} x\,\cos x\,\frac{\cos^{q+1} x}{q+1}\,dx$$

$$= -\frac{\sin^{p-1} x\cos^{q+1} x}{q+1} + \frac{p-1}{q+1}\int \sin^{p-2} x\cos^q x\,(1-\sin^2 x)\,dx$$

$$= -\frac{\sin^{p-1} x\cos^{q+1} x}{q+1} + \frac{p-1}{q+1} I_{p-2,q} - \frac{p-1}{q+1} I_{p,q}$$

Hence,

$$\frac{p+q}{q+1} I_{p,q} = -\frac{\sin^{p-1} x\cos^{q+1} x}{q+1} + \frac{p-1}{q+1} I_{p-2,q}$$

$$\Rightarrow \quad I_{p,q} = -\frac{\sin^{p-1} x\cos^{q+1} x}{p+q} + \frac{p-1}{p+q} I_{p-2,q} \quad \ldots (1)$$

Also, $$I_{p,q} = \int \sin^p x\cos^q x\, dx$$

$$= \int \cos^{q-1} x\,(\cos x \sin^p x)\, dx$$

Integrating by parts and proceeding as earlier we get,

$$I_{p,q} = \frac{\sin^{p+1} x\cos^{q-1} x}{p+q} + \frac{q-1}{p+q} I_{p,q-2} \quad \ldots (2)$$

Now we find $J_{p,q} = \int_0^{\pi/2} \sin^p x\cos^q x\, dx$, where p, q are positive integers.

From (1), we get,

$$J_{p,q} = -\left[\frac{\sin^{p-1} x\cos^{q+1} x}{p+q}\right]_0^{\pi/2} + \frac{p-1}{p+q} J_{p-2,q}$$

$$= \frac{p-1}{p+q} J_{p-2,q} \quad \ldots (3)$$

Repeating the computations in (3), we get,

$$J_{p,q} = \frac{p-1}{p+q}\,\frac{p-3}{p+q-2}\,J_{p-4,q}$$

$$= \frac{p-1}{p+q}\,\frac{p-3}{p+q-2}\,\frac{p-5}{p+q-4}\,J_{p-6,q} = \ldots$$

As a result,

$$J_{p,q} = \begin{cases} \frac{p-1}{p+q}\,\frac{p-3}{p+q-2}\cdots\frac{2}{q+3}\,J_{1,q} & \text{when } p \text{ is odd;} \\ \frac{p-1}{p+q}\,\frac{p-3}{p+q-2}\cdots\frac{1}{q+3}\,J_{0,q} & \text{when } p \text{ is even;} \end{cases}$$

Now, $$J_{1,q} = \int_0^{\pi/2} \sin x \cos^q x\,dx$$

$$= -\left[\frac{\cos^{q+1} x}{q+1}\right]_0^{\pi/2}$$

$$= \frac{1}{q+1}$$

and $$J_{0,q} = \int_0^{\pi/2} \sin^0 x \cos^q x\,dx$$

$$= \int_0^{\pi/2} \cos^q x\,dx$$

Thus when p is odd and q is any positive integer, we have,

$$J_{p,q} = \frac{p-1}{p+q}\,\frac{p-3}{p+q-2}\cdots\frac{2}{q+3}\,\frac{1}{q+1}.$$

using the reduction formula $\int_0^{\pi/2} \cos^q x\,dx$ we obtain the following. When p is even and q is odd,

$$J_{p,q} = \frac{p-1}{p+q}\,\frac{p-3}{p+q-2}\cdots\frac{1}{q+2}\times\frac{q-1}{q}\,\frac{q-3}{q-2}\cdots\frac{2}{3}$$

When p, q both are even,

$$J_{p,q} = \frac{p-1}{p+q}\,\frac{p-3}{p+q-2}\cdots\frac{1}{q+2}\times\frac{q-1}{q}\,\frac{q-3}{q-2}\cdots\frac{1}{2}\,\frac{\pi}{2}.$$

Hence, $$J_{p,q} = \begin{cases} \frac{(p-1)(p-3)\ldots(q-1)(q-3)\ldots}{(p+q)(p+q-2)\ldots}\,\frac{\pi}{2} & \text{When } p, q \text{ both are even;} \\ \frac{(p-1)(p-3)\ldots(q-1)(q-3)\ldots}{(p+q)(p+q-2)\ldots} & \text{otherwise.} \end{cases}$$

1. Remark :

One can see with the practice that when p (respectively, q) is odd, $I_{p,q}$ can be evaluated by substituting $\cos x = t$ (respectively, $\sin x = t$). In fact, the method of these substitutions works when one of p, q is odd and the other is not an integer. However, when both p and q are even, the use of reduction formula should be preferred.

Example-1 : Evaluate $\int \sin^2 x \cos^6 x\,dx$

Solution :

$$I_{2,6} = \int \sin^2 x \cos^6 x\,dx$$

$$= \frac{\sin^3 x \cos^5 x}{8} + \frac{5}{8} I_{2,4}$$

where, $$I_{2,4} = \frac{\sin^3 x \cos^3 x}{6} + \frac{3}{6} I_{2,2}$$

$$I_{2,2} = \frac{\sin^3 x \cos x}{4} + \frac{1}{4} I_{2,0}$$

$$= \frac{\sin^3 x \cos x}{4} + \frac{1}{4}\int \sin^2 x\,dx$$

$$= \frac{\sin^3 x \cos x}{4} + \frac{1}{4}\int \frac{1-\cos 2x}{2}\,dx$$

$$= \frac{\sin^3 x \cos x}{4} + \frac{1}{8}\left(x - \frac{\sin 2x}{2}\right)$$

$$= \frac{\sin^3 x \cos x}{4} + \frac{1}{8}(x - \sin x \cos x)$$

Putting all these values in $I_{2,6}$, we get,

$$I_{2,6} = \frac{\sin^3 x \cos^5 x}{8} + \frac{5\sin^3 x \cos^3 x}{48} + \frac{5\sin^3 x \cos x}{64} + \frac{5(x - \sin x \cos x)}{128}$$

Example-2 : Evaluate $\int \sin^4 x \cos^3 x\, dx$.

Solution :

Here $p = 4$ and $q = 3$, which is odd. Hence we follow the direct substitution $\sin x = t$. Then $\cos x\, dx = dt$. As a result, the given integral becomes

$$I_{4,3} = \int t^4(1-t^2)\, dt = \int (t^4 - t^6)\, dt$$

$$= \frac{t^5}{5} - \frac{t^7}{7} = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7}.$$

Example-3 : Evaluate $\int \tan^3 x \sec^3 x\, dx$

Solution :

Let $\sec x = t$. Then $\sec x \tan x\, dx = dt$. As a result, the given integral becomes

$$\int (t^2-1)t^2\, dt = \int (t^4 - t^2)\, dt = \frac{t^5}{5} - \frac{t^3}{3}$$

$$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3}$$

Example-4 : Evaluate $\int \frac{\cos^5 x}{\sin x}\, dx$

Solution :

Here $p = -1$ and $q = 5$, an odd positive integer. Hence we follow the direct substitution $\sin x = t$. Then $\cos x\, dx = dt$. As a result, the given integral becomes

$$I_{-1,5} = \int \frac{(1-t^2)^2 dt}{t}\, dt$$

$$= \int \frac{1 - 2t^2 + t^4}{t}\, dt$$

$$= \int \left(\frac{1}{t} - 2t + t^3\right) dt$$

$$= \log t - t^2 + \frac{t^4}{4}$$

$$= \log(\sin x) - \sin^2 x + \frac{\sin^4 x}{4}$$

Example-5 : Evaluate $\int_0^{\pi/2} \cos^4 x \sin 3x\, dx$

Solution :

We can write the given integral as

$$\int_0^{\pi/2} \cos^4 x(3\sin x - 4\sin^3 x)dx = 3\int_0^{\pi/2} \cos^4 x \sin x\, dx - 4\int_0^{\pi/2} \cos^4 x \sin^3 x\, dx$$

$$= 3\frac{3\cdot 1}{5\cdot 3\cdot 1} - 4\frac{3\cdot 1\cdot 2}{7\cdot 5\cdot 3} = \frac{3}{5} - \frac{8}{35} = \frac{13}{35}$$

Example-6 : Evaluate $\int_0^{\pi/4} \cos^3 2x \sin^4 4x\, dx$

Solution :

Let $2x = \theta$. Then $dx = \frac{d\theta}{2}$. Also, $x = 0 \Rightarrow \theta = 0$ and $x = \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{2}$. Hence the given integral becomes

$$\int_0^{\pi/2} \cos^3\theta \sin^4 2\theta \frac{d\theta}{2} = \frac{1}{2}\int_0^{\pi/2} \cos^3\theta(2\sin\theta\cos\theta)^4\, d\theta$$

$$= 8\int_0^{\pi/2} \cos^7\theta \sin^4\theta\, d\theta$$

$$= 8\frac{6\cdot 4\cdot 2\cdot 3\cdot 1}{11\cdot 9\cdot 7\cdot 5\cdot 3\cdot 1}$$

$$= \frac{128}{1155}.$$

Example-7 : Evaluate $\int_0^2 x^3\sqrt{2x - x^2}\, dx$

Solution :

Let $x = 2\sin^2\theta$. Then $dx = 4\sin\theta \cos\theta\, d\theta$. Also, $x = 0 \Rightarrow \theta = 0$ and $x = 2 \Rightarrow \theta = \frac{\pi}{2}$. Hence the given integral becomes

$$\int_0^{\pi/2} 8\sin^6\theta\sqrt{4\sin^2\theta - 4\sin^4\theta}\,(4\sin\theta\cos\theta)\, d\theta = 64\int_0^{\pi/2} \sin^8\theta\cos^2\theta\, d\theta$$

$$= 64\frac{7\cdot 5\cdot 3\cdot 1}{10\cdot 8\cdot 6\cdot 4\cdot 2}\frac{\pi}{2} = \frac{7\pi}{8}.$$

Example-8 : Evaluate $\int_0^1 x^{3/2}(1-x)^{3/2}\,dx$

Solution :

Let $x = \sin^2\theta$. Then $dx = 2\sin\theta\cos\theta\, d\theta$. Also, $x = 0 \Rightarrow \theta = 0$ and $x = 1 \Rightarrow \theta = \frac{\pi}{2}$. Hence the given integral becomes

$$2\int_0^{\pi/2}\sin^4\theta\cos^4\theta\, d\theta = 2\,\frac{3\cdot1\cdot3\cdot1}{8\cdot6\cdot4\cdot2}\,\frac{\pi}{2} = \frac{3\pi}{128}.$$

INTEGRAL OF $\tan^n x$ and $\cot^n x$

Now we obtain reduction formula for

$$I_n = \int \tan^n x\, dx \text{ and } J_n = \int_0^{\pi/4}\tan^n x\, dx, \text{ where } n \in N.$$

$$I_n = \int \tan^n x\, dx$$

$$= \int \tan^{n-2}x\tan^2 x\, dx$$

$$= \int \tan^{n-2}x(\sec^2 x - 1)\, dx$$

$$= \int \tan^{n-2}x\sec^2 x\, dx - \int \tan^{n-2}x\, dx$$

$$= \frac{\tan^{n-1}x}{n-1} - I_{n-2}.$$

Also, $$J_n = \int_0^{\pi/4}\tan^n x\, dx$$

$$= \left[\frac{\tan^{n-1}x}{n-1}\right]_0^{\frac{\pi}{4}} - J_{n-2}$$

$$= \frac{1}{n-1} - J_{n-2}$$

Proceeding similarly, we find

$$I_n = \int \cot^n x\, dx = -\frac{\cot^{n-1}x}{n-1} - I_{n-2} \text{ and } J_n = \int_{\pi/4}^{\pi/2}\cot^n x\, dx = \frac{1}{n-1} - J_{n-2}.$$

Example-1 : Evaluate $\int \tan^6 x\, dx$

Solution :

We know that $I_n = \int \tan^n x\, dx = \frac{\tan^{n-1}x}{n-1} - I_{n-2}$. Hence

$$I_6 = \frac{\tan^5 x}{5} - I_4, \text{ where}$$

$$I_4 = \frac{\tan^3 x}{3} - I_2, \text{ where}$$

$$I_2 = \tan x - I_0 = \tan x - \int \tan^0 x\, dx = \tan x - x.$$

Putting all these values in I_6, we get,

$$I_6 = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x.$$

Example-2 : Evaluate $\int_{\pi/4}^{\pi/2}\cot^4 x\, dx$.

Solution :

We know that $J_n = \int_{\pi/4}^{\pi/2}\cot^n x\, dx = \frac{1}{n-1} - J_{n-2}$. Hence,

$$J_4 = \frac{1}{3} - J_2 = \frac{1}{3} - [1 - J_0]$$

$$= \frac{1}{3} - 1 + J_0$$

$$= \frac{1}{3} - 1 + \int_{\pi/4}^{\pi/2}\cot^0 x\, dx$$

$$= \frac{1}{3} - 1 + \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} - \frac{2}{3}$$

$$= \frac{3\pi - 8}{12}.$$

INTEGRAL OF $\sec^n x$ and $\operatorname{cosec}^n x$

Now we obtain reduction formula $I_n = \int \sec^n x\, dx$, where $n \in N$, $n > 1$

$$I_n = \int \sec^{n-2} x \sec^2 x\, dx$$
$$= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-3} x \sec x \tan^2 x\, dx$$
$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x (\sec^2 x - 1)\, dx$$
$$= \sec^{n-2} x \tan x - (n-2)(I_n - I_{n-2})$$
$$\Rightarrow \quad I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}.$$

Similarly, we can find $I_n = \int \operatorname{cosec}^n x\, dx = -\dfrac{\operatorname{cosec}^{n-2} x \cot x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$.

Example-1 : Evaluate $\int \sec^6 x\, dx$

Solution :

We know that $I_n = \int \sec^n x\, dx = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$. Hence,

$$I_6 = \frac{\sec^4 x \tan x}{5} + \frac{4}{5} I_4$$
$$= \frac{\sec^4 x \tan x}{5} + \frac{4}{5}\left[\frac{\sec^2 x \tan x}{3} + \frac{2}{3} I_2\right]$$
$$= \frac{\sec^4 x \tan x}{5} + \frac{4}{5}\left[\frac{\sec^2 x \tan x}{3} + \frac{2}{3} \tan x\right]$$
$$= \frac{\sec^4 x \tan x}{5} + \frac{4}{15}\sec^2 x \tan x + \frac{8}{15}\tan x.$$

INTEGRAL OF $\sin^p x \cos^q x$ REVISITED

In this section we obtain the reduction formula for $I_{p,q} = \int \sin^p x \cos^q x\, dx$ where at least one of p, q is negative integer.

We know from (2), $I_{p,q} = \dfrac{\sin^{p+1} x \cos^{q-1} x}{p+q} + \dfrac{q-1}{p+q} I_{p,q-2}$

Replacing q by $q+2$ in this equation, we get,

$$I_{p,q+2} = \frac{\sin^{p+1} x \cos^{q+1} x}{p+q+2} + \frac{q+1}{p+q+2} I_{p,q}.$$

Hence,

$$I_{p,q} = -\frac{\sin^{p+1} x \cos^{q+1} x}{q+1} + \frac{p+q+2}{q+1} I_{p,q+2}, \text{ provided } q \neq -1. \quad \ldots (1)$$

The formula (1) is useful when q is a negative integer other than -1.

Similarly, we get,

$$I_{p,q} = \frac{\sin^{p+1} x \cos^{q+1} x}{p+1} + \frac{p+q+2}{p+1} I_{p+2,q}, \text{ provided } p \neq -1. \quad \ldots (2)$$

The formula (2) is useful when p is a negative integer other than -1.

Example-1 : Evaluate $\int \sin^2 x \cos^{-3} x\, dx$.

Solution :

Here q is a negative integer other than -1, so,

$$I_{p,q} = -\frac{\sin^{p+1} x \cos^{q+1} x}{q+1} + \frac{p+q+2}{q+1} I_{p,q+2}$$

Hence,

$$I_{2,-3} = -\frac{\sin^3 x \cos^{-2} x}{-2} + \frac{1}{-2} I_{2,-1}$$
$$= \frac{\sin^3 x \cos^{-2} x}{2} - \frac{1}{2}\int \sin^2 x \cos^{-1} x\, dx$$
$$= \frac{\sin^3 x \cos^{-2} x}{2} - \frac{1}{2}\int \frac{1-\cos^2 x}{\cos x}\, dx$$
$$= \frac{\sin^3 x \cos^{-2} x}{2} - \frac{1}{2}\int (\sec x - \cos x)\, dx$$
$$= \frac{\sin^3 x \cos^{-2} x}{2} - \frac{1}{2}\log(\sec x + \tan x) + \frac{1}{2}\sin x$$
$$= \frac{1}{2}\sin x \tan^2 x - \frac{1}{2}\log(\sec x + \tan x) + \frac{1}{2}\sin x$$
$$= \frac{1}{2}\sin x(\tan^2 x + 1) - \frac{1}{2}\log(\sec x + \tan x)$$
$$= \frac{1}{2}\sin x \sec^2 x - \frac{1}{2}\log(\sec x + \tan x).$$

EXERCISE

Evaluate the following :

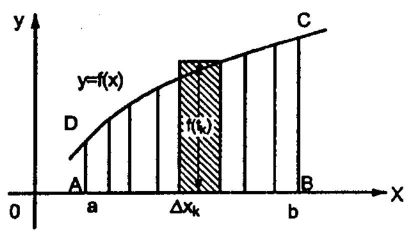
1. $\int \sin^7 x\, dx$
2. $\int \cos^{11} x\, dx$
3. $\int \sin^{10} x\, dx$
4. $\int \dfrac{\sin^6 x}{\cos x}\, dx$
5. $\int \dfrac{1}{\sin^2 x \cos^4 x}\, dx$
6. $\int \dfrac{1}{\sin^{3/2} x \cos^{5/2} x}\, dx$
7. $\int \dfrac{(1-\cos x)^{3/5}}{(1+\cos x)^{8/5}}\, dx$
8. $\int \sec^{4/7} x \operatorname{cosec}^{10/7} x\, dx$
9. $\int \sec^{8/9} x \operatorname{cosec}^{10/9} x\, dx$
10. $\int \dfrac{\sin^3 x}{\cos^4 x}\, dx$
11. $\int \sin^4 x \cos^{-5} x\, dx$
12. $\int \sin^{-4} x \cos^5 x\, dx$
13. $\int \sin^2 x \cos^6 x\, dx$
14. $\int \sin^4 x \cos^4 x\, dx$
15. $\int \sec x \tan^5 x\, dx$
16. $\int \cot^5 x \cos^5 x\, dx$
17. $\int \sin^4 x \cos^2 x\, dx$
18. $\int \tan^2 x \sec x\, dx$
19. $\int \dfrac{dx}{(a^2+b^2x^2)^{3/2}}$
20. $\int \dfrac{dx}{(a^2x^2-b^2)^{3/2}}$
21. $\int \operatorname{cosec}^5 x\, dx$
22. $\int \sec^4 x\, dx$
23. $\int \sec^5 x\, dx$
24. $\int \cot^5 x\, dx$
25. $\int \tan^4 x\, dx$
26. $\int_0^{\pi/4} \tan^6 x\, dx$
27. $\int_{\pi/4}^{\pi/2} \cot^6 x\, dx$
28. $\int_0^{\pi/2} \cos^{12} x\, dx$
29. $\int_0^{\pi/2} \sin^8 x\, dx$
30. $\int_0^{\pi/6} \sin^6 3x\, dx$
31. $\int_0^{\pi/4} \cos^6 2x\, dx$
32. $\int_0^1 x^6 \sqrt{1-x^2}\, dx$
33. $\int_0^{2a} x^3 (2ax-x^2)^{3/2}\, dx$
34. $\int_0^{\pi/4} \cos^{3/2} 2x \cos x\, dx$
35. $\int_0^{\pi/6} \cos^4 3x \sin^2 6x\, dx$

36. $\int_0^{\pi} \dfrac{\sin^4\theta}{(1+\cos\theta)^2}\, d\theta$
37. $\int_0^{\pi} \dfrac{\sin^2 x\sqrt{1-\cos x}}{1+\cos x}\, dx$
38. $\int_0^1 \dfrac{x^6}{\sqrt{1-x^2}}\, dx$
39. $\int_0^{\infty} \dfrac{1}{(1+x^2)^{7/2}}\, dx$
40. $\int_0^1 \dfrac{x^2(2-x^2)}{\sqrt{1-x^2}}\, dx$
41. $\int_0^2 x^2(4-x^2)^{9/2}\, dx$
42. $\int_0^1 \dfrac{x^7}{\sqrt{1-x^4}}\, dx$
43. $\int_0^1 x^5 \sqrt{\dfrac{1+x^2}{1-x^2}}\, dx$
44. $\int_0^{\infty} \dfrac{x^2}{\sqrt{(1+x^6)^7}}\, dx$
45. $\int_0^1 x^6 \sin^{-1} x\, dx$
46. $\int_0^{\pi/2} \sin^{3/2} x \cos^3 x\, dx$
47. $\int_0^{\pi/2} \sin^8 x \cos^{10} x\, dx$
48. $\int_0^{\pi/2} \sin 3x \cos^4 x\, dx$
49. $\int_0^{\pi/2} \sin^4 4x \cos^3 2x\, dx$
50. $\int_0^{\infty} \dfrac{x^3}{(1+x^2)^{9/2}}\, dx$
51. $\int_0^{\infty} \dfrac{x^2}{(1+x^2)^{7/2}}\, dx$
52. $\int_0^2 x^{5/2}\sqrt{2-x}\, dx$
53. $\int_0^{2a} x^{9/2}(2a-x)^{-1/2}\, dx$
54. $\int_0^{2a} x^2\sqrt{2ax-x^2}\, dx$
55. $\int_0^{\infty} \dfrac{x^3}{(4+x^2)^3}\, dx$
56. $\int_0^1 x^{3/2}(1-x)^{3/2}\, dx$
57. $\int \sin^5 x \cos^{-4} x\, dx$. (Express the answer in terms of cosine function.)
58. $\int \cos^5 x \sin^{-1} x\, dx$. (Express the answer in terms of sine function.)

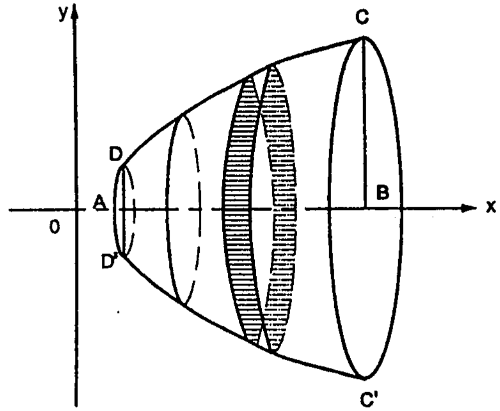
VOLUME OF A SOLID OF REVOLUTION

VOLUME OF A SOLID OF REVOLUTION BY CIRCULAR DISKS

Let $y = f(x)$ be a continuous function such that $f(x) \geq 0$ on $[a, b]$. Then the curve $y = f(x)$, the lines $x = a$, $x = b$, and the x-axis form the boundary of the area ABCD (Fig. 1(a)). If this area is revolved about the x-axis, we say that it generates a solid of revolution (Fig. 2(b)).



(a)



(b)

Our problem now is to define the volume of a solid of revolution and see how to compute the volume. We divide the interval $[a, b]$ into n subintevals, and on each subinterval as a base draw a rectangle, the altitude being the ordinate of the curve drawn from some point of the base. The widths of the rectangles are $\Delta x_1, \Delta x_2, \ldots \Delta x_n$, and the altitudes are $f(t_1), f(t_2), \ldots, f(t_n)$. Now suppose these rectangles are revolved about the x-axis. Each generates a right circular cylinder, or a disk. The volume of a right circular cylinder is $\pi r^2 h$, where r is the radius and h is the altitude. Then the sum of the volumes of the n cylinder is

$$V_n = \sum_{k=1}^{n} \pi[f(t_k)]^2 \Delta x_k .$$

If all the Δx's are quite small, we see intuitively that the solid swept out by the rectangles almost coincides with the volume swept out by the area under the curve. Hence we are led to define the volume of the solid of revolution of the area ABCD as the limit of V_n as $n \to \infty$ and all the Δx's $\to 0$. We then have

$$V = \pi\int_a^b [f(x)]^2\, dx, \quad \text{or} \quad V = \pi\int_a^b y^2\, dx \qquad \ldots (1)$$

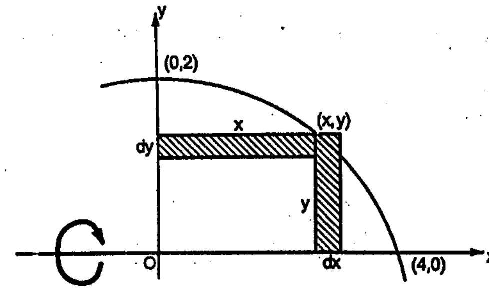
Similarly, the area bounded by $x = f(y)$, $y = c$, $y = d$, and the y-axis, when revolved about the y-axis, would sweep out a solid whose volume is given by

$$V = \pi\int_c^d [f(y)]^2\, dy, \quad \text{or} \quad V = \pi\int_c^d x^2\, dy \qquad \ldots (2)$$

Example-1 :

The area bounded by the curve $y = \sqrt{4 - x}$, the line $x = 0$ and the line $y = 0$ is revolved about the x-axis. Find the volume of the solid thus generated. Also find the volume if the area is revolved about the y-axis.

Solution :



Volume $V = \pi\int_a^b y^2 dx$

$$= \pi\int_0^4 (4 - x)dx$$

$$= \pi\left[\frac{(4-x)^2}{2}(-1)\right]_0^4$$

$$= -\frac{\pi}{2}[0 - 16]$$

$$= 8\pi$$

Also volume of the solid made by revolving the area about the y-axis is

$$V = \pi\int_c^d x^2 dy$$

$$= \pi\int_0^2 (4 - y^2)^2 dy \qquad [\because y^2 = 4 - x \Rightarrow x = 4 - y^2]$$

$$= \pi\int_0^2 [16 - 8y^2 + y^4]\, dy$$

$$= \pi\left[16y - \frac{8y^3}{3} + \frac{y^5}{5}\right]_0^2$$

$$= \pi\left[32 - \frac{64}{3} + \frac{32}{5}\right]$$

$$= \frac{\pi}{15}[480 - 320 + 96]$$

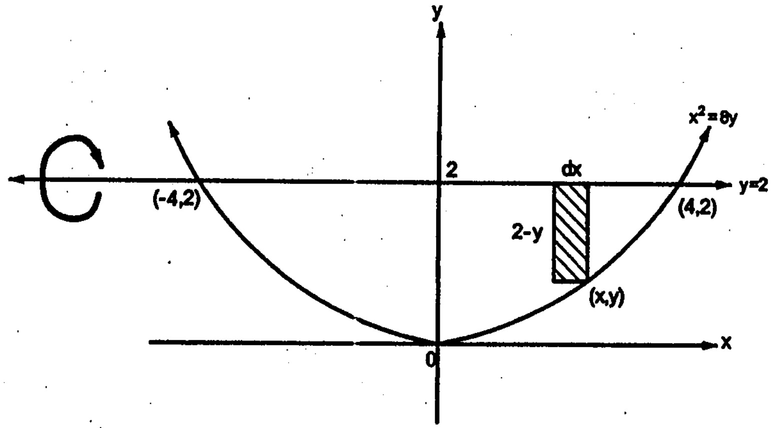
$$= \frac{256\pi}{15}.$$

Example-2 :

The area bounded by the parabola $x^2 = 8y$ and the line $y = 2$ is revolved about the line $y = 2$. Find the volume of the solid thus generated.

Solution :

From figure we say that the altitude of the rectangle is $2 - y$.



Hence the element of volume is $dV = \pi(2-y)^2 dx$

$\therefore$ Required volume is

$$V = \int_a^b dV = \int_{-4}^{4} \pi(2-y)^2 dx$$

$$= 2\pi \int_0^4 \left(2 - \frac{x^2}{8}\right)^2 dx \qquad \left[\because\ y = \frac{x^2}{8}\right]$$

$$= 2\pi \int_0^4 \left[4 - \frac{x^2}{2} + \frac{x^4}{64}\right] dx$$

$$= 2\pi \left[4x - \frac{x^3}{6} + \frac{x^5}{320}\right]_0^4$$

$$= 2\pi \left[16 - \frac{64}{6} + \frac{4^5}{320}\right]$$

$$= 2\pi \left[16 - \frac{32}{3} + \frac{16}{5}\right]$$

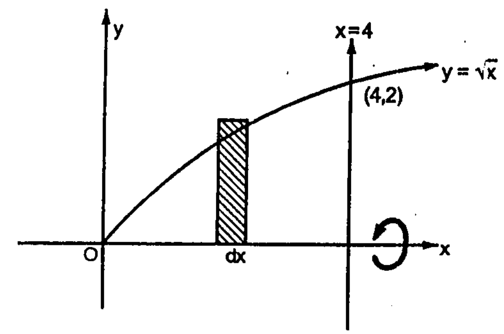
$$= \frac{2\pi}{15}[240 - 160 + 48]$$

$$= \frac{256\pi}{15}.$$

Example-3 :

The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$ and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

Solution :



Required volume

$$V = \pi \int_a^b y^2\, dx = \pi \int_0^4 x\, dx$$

$$= \pi \left[\frac{x^2}{2}\right]_0^4$$

$$= \frac{\pi}{2}[16 - 0]$$

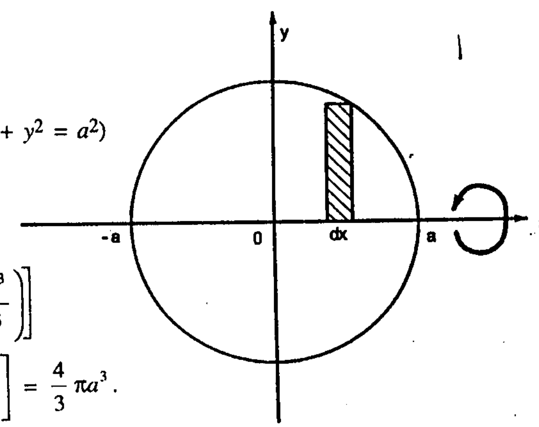
$$= 8\pi.$$

Example-4 :

The circle $x^2 + y^2 = a^2$ is rotated about the x-axis to generate a sphere. Find its volume.

Solution :

Required volume



$$V = \pi \int_a^b y^2\, dx$$

$$= \pi \int_{-a}^{a} (a^2 - x^2)\, dx \qquad (\because\ x^2 + y^2 = a^2)$$

$$= \pi \left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a}$$

$$= \pi \left[\left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 + \frac{a^3}{3}\right)\right]$$

$$= \pi \left[2a^3 - \frac{2a^3}{3}\right] = \pi \left[\frac{4a^3}{3}\right] = \frac{4}{3}\pi a^3.$$

Example-5 :

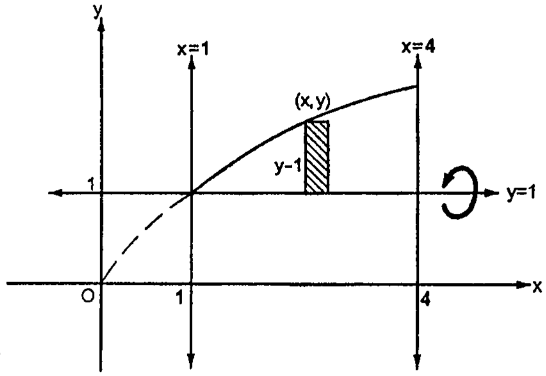
Find the volume of the solid generated by revolving the region bounded by $y=\sqrt{x}$ and the line $y = 1$, $x = 4$ about the line $y = 1$.

Solution :

From figure we say that the altitude of the rectangle is $y - 1$.

Hence the element of volume is $dV = \pi(y - 1)^2\, dx$

$\therefore$ Required volume



$$V = \int_a^b dV$$

$$= \int_{x=1}^{4} \pi(y-1)^2\, dx$$

$$= \int_1^4 \pi\,[\sqrt{x}-1]^2\, dx$$

$$= \pi\int_1^4 [x - 2\sqrt{x} + 1]dx$$

$$= \pi\left[\frac{x^2}{2} - \frac{4x^{3/2}}{3} + x\right]_1^4$$

$$= \pi\left[\left(8 - \frac{32}{3} + 4\right) - \left(\frac{1}{2} - \frac{4}{3} + 1\right)\right]$$

$$= \pi\left[\left(12 - \frac{32}{3}\right) - \left(\frac{3}{2} - \frac{4}{3}\right)\right]$$

$$= \pi\left[\frac{4}{3} - \frac{1}{6}\right]$$

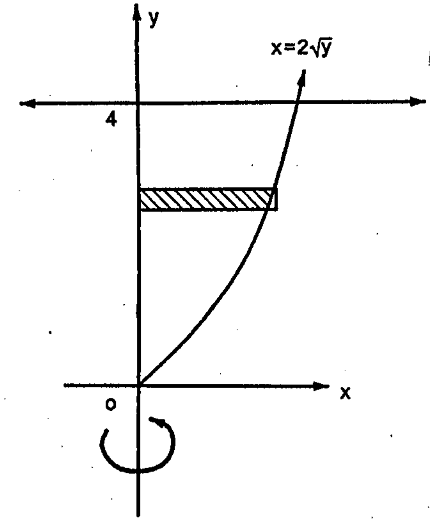
$$= \pi\left[\frac{7}{6}\right]$$

$$= \frac{7\pi}{6}$$

Example-6 :

Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2\sqrt{y}$, $0 \le y \le 4$, about the y-axis.

Solution :



$$V = \pi\int_c^d x^2 dy$$

$$= \pi\int_0^4 4y\, dy$$

$$= 4\pi\left[\frac{y^2}{2}\right]_0^4$$

$$= 4\pi\ [8]$$

$$= 32\pi.$$

Example-7 :

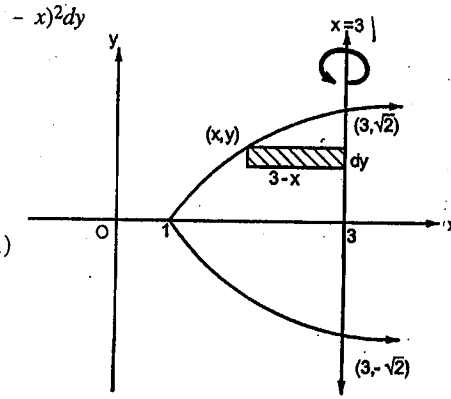
Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$, about the line $x = 3$.

Solution :

From figure we say that $-\sqrt{2} \le y \le \sqrt{2}$

From figure we say that the altitude of the rectangle is $3 - x$.

Hence the element of volume is $dV = \pi(3 - x)^2 dy$



$$\therefore\quad V = \int dV$$

$$= \int_c^d \pi(3-x)^2 dy$$

$$= \pi\int_{-\sqrt{2}}^{\sqrt{2}} [2 - y^2]^2\, dy \quad (\because\ x = y^2 + 1)$$

$$= 2\pi\int_0^{\sqrt{2}} [4 - 4y^2 + y^4]\, dy$$

$$= 2\pi\left[4y - \frac{4y^3}{3} + \frac{y^5}{5}\right]_0^{\sqrt{2}}$$

$$= 2\pi\left[4\sqrt{2} - \frac{8\sqrt{2}}{3} + \frac{4\sqrt{2}}{5}\right]$$

$$= 4\sqrt{2}\cdot 2\pi\left[1 - \frac{2}{3} + \frac{1}{5}\right]$$

$$= 8\sqrt{2}\,\pi\left[\frac{15-10+3}{15}\right]$$

$$= \frac{64\sqrt{2}\pi}{15}$$

EXERCISE

In each of Problems 1–17 find the volume of the solid generated when the area bounded by the given curves is revolved about the given line.

1. $y = x$, $y = 0$, $x = 3$; revolve about the x-axis.

2. $y = x$, $x = 0$, $y = 3$; revolve about the y-axis.

3. $x + y = 4$, $x = 0$, $y = 0$; revolve about the x-axis.

4. $x + y = 4$, $x = 0$, $y = 0$; revolve about the y-axis.

5. $y^2 = 4x$, $x = 4$; revolve about the x-axis.

6. $y^2 = 4x$, $y = 4$, $x = 0$; revolve about the y-axis.

7. $y = 1 - x^2$, $y = 0$; revolve about the x-axis.

8. $y = 1 - x^2$, $y = -1$; revolve about the y-axis.

9. $y = x$, $y = 0$, $x = 3$; revolve about $x = 3$.

10. $y = x$, $x = 0$, $y = 3$; revolve about $y = 3$.

11. $y^2 = 4x$, $x = 4$; revolve about $x = 4$.

12. $y^2 = 4x$, $y = 4$, $x = 0$; revolve about $y = 4$.

13. $y = 1 - x^2$, $y = -1$; revolve about $y = -1$.

14. $y = x^2 - 2x$, $y = 0$; revolve about the x-axis.

15. $y = x^2 - 2x - 4$, $y = 0$; revolve about $x = 1$.

16. $y = 4x - x^2$, $y = 0$; revolve about the x-axis.

17. $y = 3x - x^2$, $y = 0$; revolve about $x = 2$.

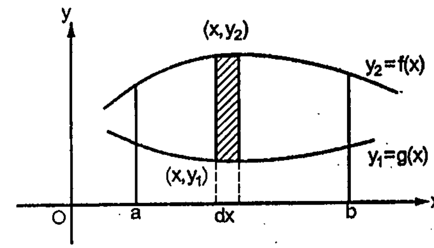
VOLUME BY THE WASHER METHOD

Let $y_2 = f(x)$ and $y_1 = g(x)$ be continuous functions such that $f(x) > g(x) > 0$ on the interval $[a, b]$. Then let the area bounded by the curves and the lines $x = a$ and $x = b$ be revolved about the x-axis. A hollow solid of revolution is swept out. Clearly the volume of the solid is the volume generated by the area under the upper curve minus the volume generated by the area under the lower curve. That is,

$$V = \pi\int_a^b y_2^2\,dx - \pi\int_a^b y_1^2\,dx$$

or, combining the integrals,

$$V = \pi\int_a^b (y_2^2 - y_1^2)\,dx \qquad \dots (1)$$



We could think of Formula (1) as being the limit of the sum of volumes of washer-shaped solids. Referring to Fig. 9, we see that a typical solid is generated by revolving the shaded rectangle about the x-axis. The volume swept out by this rectangle is given by

$$dV = \pi y_2^2 dx - \pi y_1^2 dx = \pi(y_2^2 - y_1^2)\,dx,$$

and this is the element of volume appearing in Formula (1).

Example-1 :

Find the volume generated by revolving about the x-axis the area bounded by the parabolas $x^2 = 2y$ and $x^2 = 12 - 4y$.

Solution :

First we find point of intersection $x^2 = 2y$ and $x^2 = 12 - 4y$.

$\Rightarrow$ $2y = 12 - 4y$

$\Rightarrow$ $6y = 12$

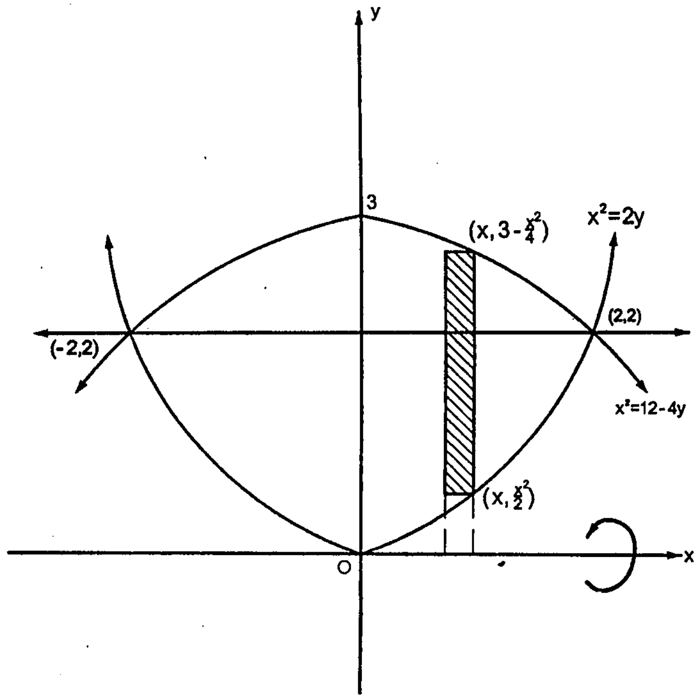
$\Rightarrow$ $y = 2$

$\therefore$ $x^2 = 2y = 4$

$\Rightarrow$ $x = \pm 2$

$\therefore$ $(-2, 2)$ and $(2, 2)$ are point of intersection

Also $-2 \le x \le 2$



From figure we say that upper curve is $x^2 = 12 - 4y$ i.e. $y = 3 - \frac{x^2}{4}$

and lower curve is $x^2 = 2y$ i.e. $y = \frac{x^2}{2}$

$$\therefore \quad V = \pi \int_a^b [y_2^2 - y_1^2]\, dx$$

$$\therefore \quad V = \pi \int_{-2}^{2} \left[\left(3 - \frac{x^2}{4}\right)^2 - \left(\frac{x^2}{2}\right)^2\right] dx$$

$$= 2\pi \int_0^2 \left[9 - \frac{3x^2}{2} + \frac{x^4}{16} - \frac{x^4}{4}\right] dx$$

$$= 2\pi \int_0^2 \left[9 - \frac{3x^2}{2} - \frac{3x^4}{16}\right] dx$$

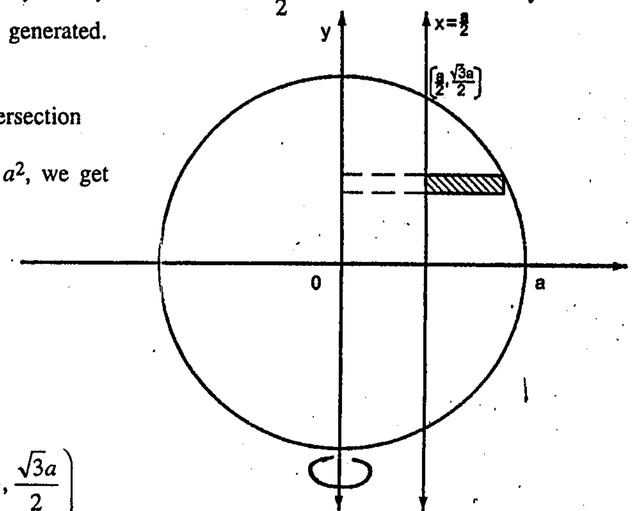
$$= 2\pi \left[9x - \frac{x^3}{2} - \frac{3x^5}{80}\right]_0^2$$

$$= 2\pi \left[18 - 4 - \frac{96}{80}\right]$$

$$= 2\pi \left[14 - \frac{6}{5}\right] = 2\pi \left[\frac{64}{5}\right] = \frac{128\pi}{5}$$

Example-2 :

The smaller area bounded by $x^2 + y^2 = a^2$ and $x = \frac{a}{2}$ is revolved about the y-axis. Find the volume of the solid thus generated.



Solution :

First we find pint of intersection

Put $x = \frac{a}{2}$ in $x^2 + y^2 = a^2$, we get

$$\frac{a^2}{4} + y^2 = a^2$$

$$\Rightarrow \quad y^2 = a^2 - \frac{a^2}{4} = \frac{3a^2}{4}$$

$$\Rightarrow \quad y = \frac{\pm\sqrt{3}a}{2}$$

$$\therefore \quad \left(\frac{a}{2}, \frac{-\sqrt{3}a}{2}\right) \text{ and } \left(\frac{a}{2}, \frac{\sqrt{3}a}{2}\right)$$

are point of intersection

Also $\frac{-\sqrt{3}a}{2} \le y \le \frac{\sqrt{3}a}{2}$

From figure we say that upper curve is $x^2 + y^2 = a^2$ i.e. $x = \pm\sqrt{a^2 - y^2}$

and lower curve is $x = \frac{a}{2}$

$\therefore \quad V = \pi \int_c^d (x_2^2 - x_1^2)\, dy$

$$= \pi \int_{-\frac{\sqrt{3}a}{2}}^{\frac{\sqrt{3}a}{2}} \left[(a^2 - y^2) - \frac{a^2}{4}\right] dy$$

$$= 2\pi \int_0^{\frac{\sqrt{3}a}{2}} \left[\frac{3a^2}{4} - y^2\right] dy$$

$$= 2\pi \left[\frac{3a^2}{4} y - \frac{y^3}{3}\right]_0^{\frac{\sqrt{3}a}{2}}$$

$$= 2\pi \left[\frac{3\sqrt{3}a^3}{8} - \frac{3\sqrt{3}a^3}{24}\right]$$

$$= 2\pi a^3 \sqrt{3}\left[\frac{3}{8} - \frac{1}{8}\right]$$

$$= 2\pi\sqrt{3}a^3 \left[\frac{2}{8}\right]$$

$$= \frac{\sqrt{3}\pi a^3}{2}$$

Example-3 :

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution :

By comparing $y = x^2 + 1$ and $y = -x + 3$

we get $x^2 + 1 = -x + 3$

$\Rightarrow \quad x^2 + x - 2 = 0$

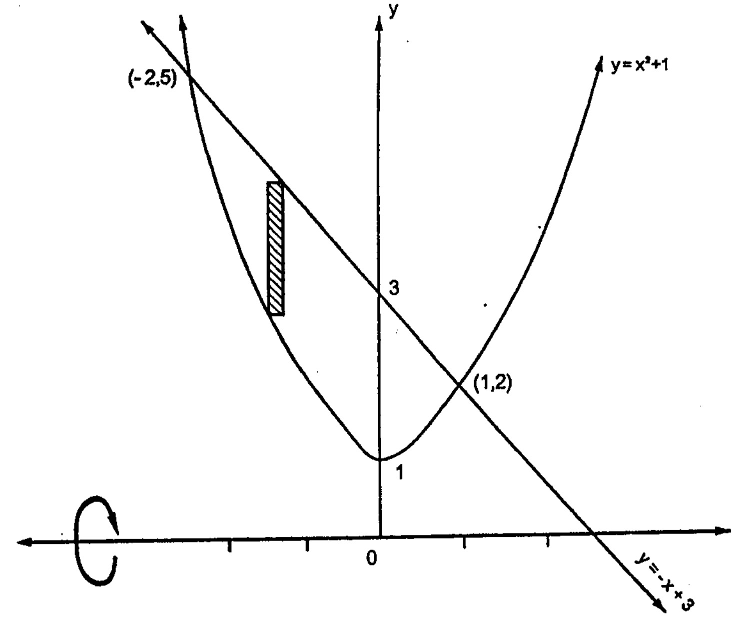
$\Rightarrow \quad (x + 2)(x - 1) = 0$

$\Rightarrow \quad x = -2, x = 1$

For $x = -2$, $y = -x + 3 = 2 + 3 = 5$

For $x = 1$, $y = -x + 3 = -1 + 3 = 2$

$\therefore$ Point of intersection are $(-2, 5)$ and $(1, 2)$.



From figure we say that $-2 \le x \le 1$, outer curve is $y = -x + 3$ and inner curve is $y = x^2 + 1$.

$\therefore \quad V = \pi \int_a^b [y_2^2 - y_1^2]\, dx$

$$= \pi \int_{-2}^{1} [(-x + 3)^2 - (x^2 + 1)^2]\, dx$$

$$= \pi \int_{-2}^{1} [x^2 - 6x + 9 - x^4 - 2x^2 - 1]\, dx$$

$$= \pi \int_{-2}^{1} [-x^2 - 6x - x^4 + 8]\, dx$$

$$= \pi \left[\frac{-x^3}{3} - 3x^2 - \frac{x^5}{5} + 8x\right]_{-2}^{1}$$

$$= \pi \left[\left(\frac{-1}{3} - 3 - \frac{1}{5} + 8\right) - \left(\frac{8}{3} - 12 + \frac{32}{5} - 16\right)\right]$$

$$= \pi\left[\frac{-1}{3} - \frac{1}{5} + 5 - \frac{8}{3} - \frac{32}{5} + 28\right]$$

$$= \pi\left[-3 - \frac{33}{5} + 33\right]$$

$$= \pi\left[30 - \frac{33}{5}\right]$$

$$= \pi\left[\frac{150 - 33}{5}\right] = \frac{117\pi}{5}$$

Example-4 :

The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution :

By comparing $y = x^2$ and $y = 2x$ we get $x^2 = 2x$

$\Rightarrow \quad x^2 - 2x = 0$

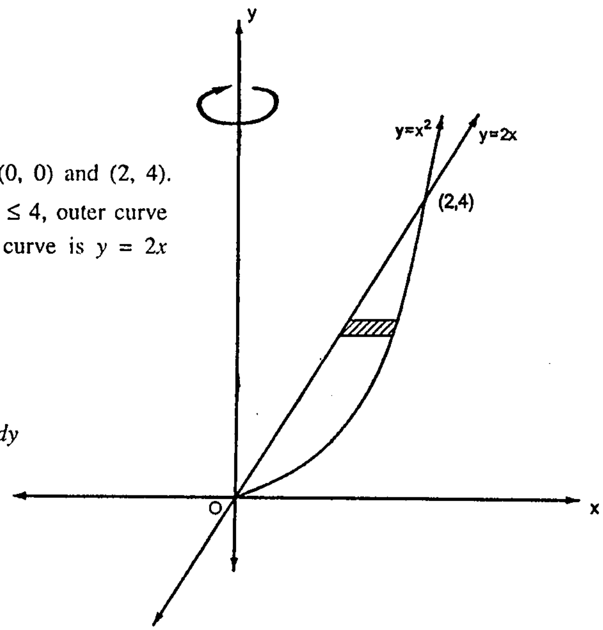
$\Rightarrow \quad x(x - 2) = 0$

$\Rightarrow \quad x = 0, 2$

For $x = 0$, $y = 2x = 0$

For $x = 2$, $y = 2x = 4$

$\therefore$ Point of intersection are (0, 0) and (2, 4).



From figure we say that $0 \le y \le 4$, outer curve is $y = x^2$ i.e. $x = \sqrt{y}$ and inner curve is $y = 2x$ i.e. $x = \frac{y}{2}$.

$$\therefore \quad V = \pi\int_c^d (x_2^2 - x_1^2)\,dy$$

$$= \pi\int_0^4 \left[(\sqrt{y}) - \left(\frac{y}{2}\right)^2\right] dy$$

$$= \pi\int_0^4 \left[y - \frac{y^2}{4}\right] dy$$

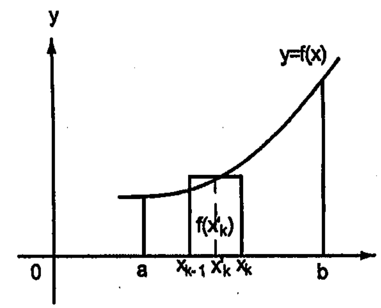
$$= \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4$$

$$= \pi\left[8 - \frac{16}{3}\right] = \pi\left[\frac{24 - 16}{3}\right] = \frac{8\pi}{3}$$

VOLUME BY THE CYLINDRICAL SHELL METHOD

It often happens that the volume of a solid of revolution can be found more easily by using cylindrical shells. The element of volume in this case is a thin hollow cylinder which is generated by revolving a rectangle with a narrow base about a line parallel to its altitude.

Suppose the area bounded by $y = f(x)$, $y = 0$, $x = a$ and $x = b$ (Fig. 14) is revolved about the y-axis. Imagine now that n narrow rectangles are constructed from a to b such as the k^{th} rectangle in the figure. The altitude of the rectangle is the ordinate of the curve at some point x'_k of the k^{th} subinterval and the base is equal to $x_k - x_{k-1} = \Delta x_k$. When this rectangle is revolved about the y-axis, a hollow shell is swept out. Since the outer radius of the shell is x_k and the inner radius is x_{k-1}, the volume ΔV_k of the shell is given by



$$\Delta V_k = \pi(x_k^2 - x_{k-1}^2)\, f(x'_k)$$

$$= \pi(x_k + x_{k-1})(x_k - x_{k-1})\, f(x'_k)$$

$$= 2\pi\frac{(x_k + x_{k-1})}{2} f(x'_k)\,\Delta x_k$$

$$= 2\pi x''_k\, f(x'_k)\,\Delta x_k$$

where, $x''_k = \frac{1}{2}(x_k + x_{k-1})$, the midpoint of the k^{th} subinterval. Hence the volume swept out by the n rectangles would be

$$V_n = \sum_{k=1}^{n} 2\pi x''_k\, f(x'_k)\,\Delta x_k.$$

It seems reasonable to call the limit of V_n as $n \to \infty$ and each $\Delta x_k \to 0$ the volume swept out by the area under the curve from $x = a$ to $x = b$. Now if x''_k and x'_k were the same, we would express the limit by

$$V = 2\pi\int_a^b x\, f(x)\,dx \quad \text{or} \quad V = 2\pi\int_a^b xy\,dx \qquad \ldots (1)$$

We cannot conclude, on the basis of our previous discussions, that this integral actually gives the limit. Its correctness follows, however, from a theorem due to the American mathematician G.A. Bliss (1876–1950), which we quote.

The Bliss Theorem. Let f and g be continuous functions on the interval $[a, b]$. Let the interval be divided into n subintervals and denote two arbitrary points of the k^{th} subinterval

(k = 1, 2, 3, ..., n) by x_k' and x_k''. Let Δx_k be the distance between the endpoints x_{k-1} and x_k of the k^{th} subinterval. Then if n increases indefinitely and each $\Delta x_k \to 0$, it follows that

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k')\, g(x_k'')\, \Delta x_k = \int_a^b f(x)\, g(x)\, dx.$$

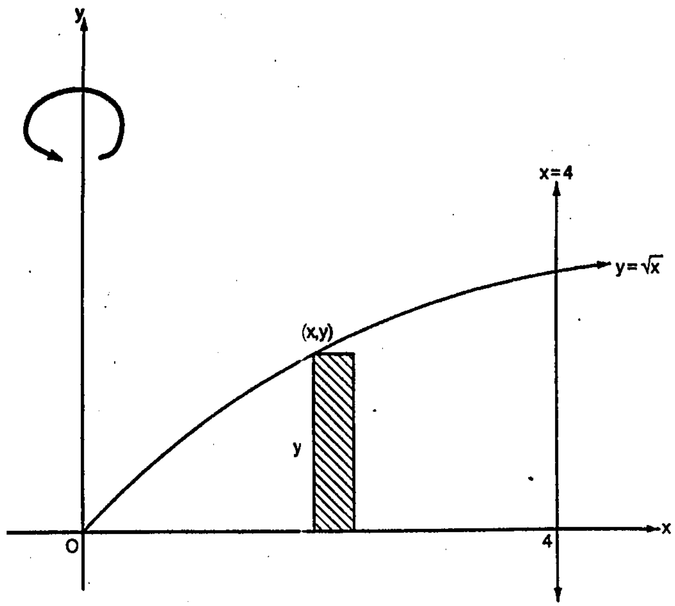
In previous Section we defined the volume of a solid of revolution as the limit of a sum of circular cylinders, and derived the formula for the volume. We now have another formulae obtained by the use of cylindrical shells, which seems, to yield the volume. We have not shown that the two methods are equivalent. It can be proved, however, that the application of the two methods to the same problem will give the same result. We shall assume this to be the case.

Coming back to Formula (1), we observe that $2\pi x$ represents the circumference, y the altitude, and dx the thickness of a cylindrical shell. Hence a typical element of volume is equal to the *circumference of the shell times the height times the thickness*. From this fact, we see that the shell method is adaptable when the area is revolved about the x-axis or some other line. Accordingly, it is the idea embodied in the formula, rather than the arrangement of letters, which is worth remembering.

Example-1 :

The region bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid by shell method.

Solution :



From figure we say that $0 \le x \le 4$

$$V = 2\pi \int_a^b xy\, dx$$

$$= 2\pi \int_0^4 x\sqrt{x}\, dx \qquad (\because\ y = \sqrt{x})$$

$$= 2\pi \left[\frac{x^{5/2}}{5/2}\right]_0^4 = \frac{4\pi}{5}[4^{5/2}] = \frac{4\pi}{5}[2^5] = \frac{128\pi}{5}$$

Example-2 :

The region bounded by the curve $y = \sqrt{4x - x^2}$, the x-axis and the line $x = 2$ is revolved about the x-axis. Find the volume of the solid by cylindrical shell.

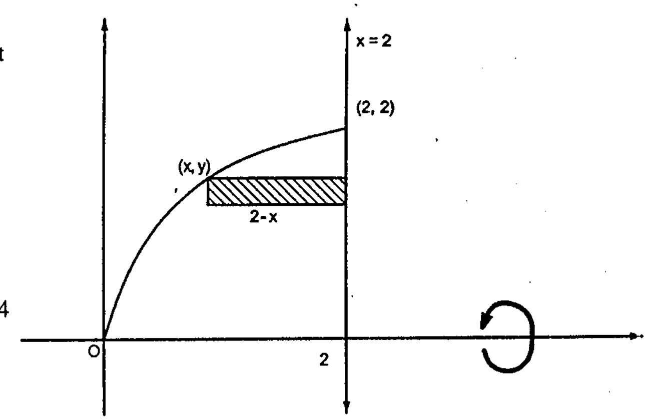
Solution :

From figure we say that

Shell height = $2 - x$

Shell radius = y

and $0 \le y \le 2$



Also $y = \sqrt{4x - x^2}$

$\Rightarrow\ y^2 = 4x - x^2$

$\Rightarrow\ x^2 - 4x + 4 + y^2 = 4$

$\Rightarrow\ (x-2)^2 + y^2 = 4$

$\Rightarrow\ (2-x)^2 + y^2 = 4$

$\Rightarrow\ 2 - x = \sqrt{4 - y^2}$

$$V = 2\pi \int_a^b (\text{shell radius})\,(\text{shell height})\, dy$$

$$= 2\pi \int_0^2 y\,(2 - x)\, dy$$

$$= 2\pi \int_0^2 y\sqrt{4 - y^2}\, dy$$

Let $4 - y^2 = t^2$ then $-2y\, dy = 2t\, dt$

$y\, dy = -t\, dt$

For $y = 0$, $t = 2$ and $y = 2$, $t = 0$

$$\therefore \quad V = 2\pi \int_2^0 t(-t\,dt)$$

$$= 2\pi \int_0^2 t^2\,dt$$

$$= 2\pi \left[\frac{t^3}{3}\right]_0^2$$

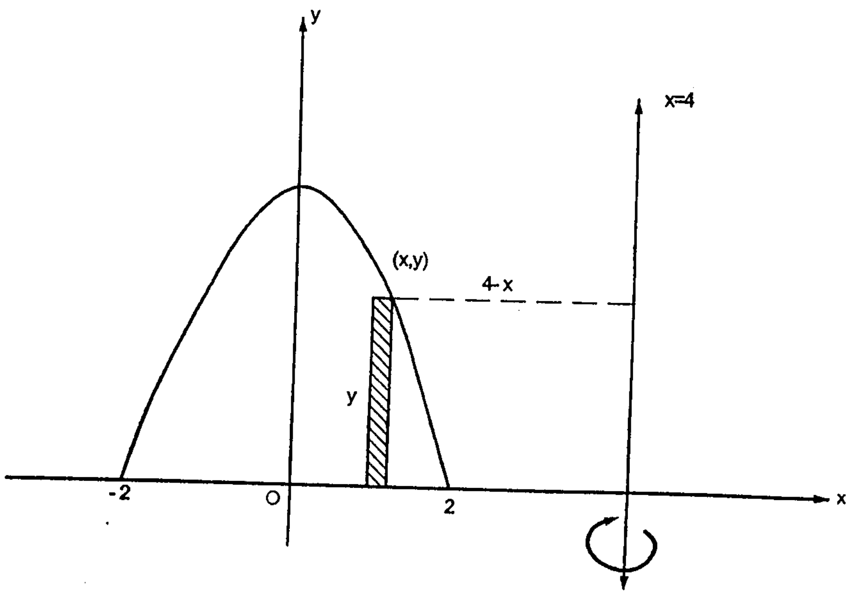
$$= 2\pi \left[\frac{8}{3}\right]$$

$$= \frac{16\pi}{3}$$

Example-3 :

The area bounded by $x^2 = 4 - y$ and x-axis is revolved about the line $x = 4$. Find the volume of the solid thus generated.

Solution :



From figure we say that

Shell radius = $4 - x$

Shell height = y

Also $-2 \le x \le 2$

$$\therefore \quad V = 2\pi \int_c^d (\text{shell radius})(\text{shell height})\,dx$$

$$= 2\pi \int_{-2}^{2} (4-x)\,y\,dx$$

$$= 2\pi \int_{-2}^{2} (4-x)(4-x^2)\,dx \qquad (\because x^2 = 4 - y)$$

$$= 2\pi \int_{-2}^{2} (16 - 4x - 4x^2 + x^3)\,dx$$

$$= 2\pi \left[16x - 2x^2 - \frac{4x^3}{3} + \frac{x^4}{4}\right]_{-2}^{2}$$

$$= 2\pi \left[\left(32 - 8 - \frac{32}{3} + 4\right) - \left(-32 - 8 + \frac{32}{3} + 4\right)\right]$$

$$= 2\pi \left[64 - \frac{64}{3}\right]$$

$$= 128\pi \left[1 - \frac{1}{3}\right]$$

$$= 128\pi \left[\frac{2}{3}\right]$$

$$= \frac{512\pi}{3}$$

EXERCISE

Use the washer method to find the volume of the solid generated when the area bounded by each of the given curves is revolved about the given line.

1. $y = x$, $y = 0$, $x = 3$; revolve about the y-axis.
2. $y = x$, $y = 0$, $x = 3$; revolve about $y = 3$.

3. $x + y = 4$, $x = 0$, $y = 0$; revolve about $x = 4$.
4. $y^2 = 4x$, $x = 4$; revolve about the y-axis.
5. $y^2 = 4x$, $y = 4$, $x = 0$; revolve about the x-axis.
6. Use the cylindrical shell method to find each of the volumes of Problem 1–5 above.

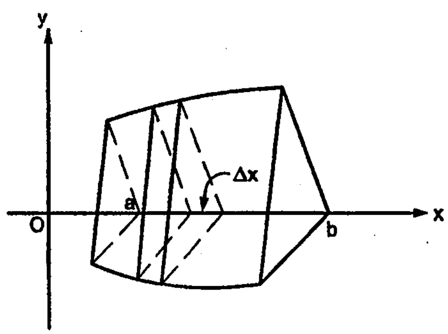
Find the volume defined in each of Problems 7–17 by either the disc, washer, or shell method.

7. $y = x^2 - 2x$, $y = 0$; revolve about $x = 2$.
8. $y = 4x - x^2$, $y = 0$; revolve about the y-axis.
9. $y = 4x - x^2$, $y = 0$; revolve about $x = 6$.
10. $y^2 = x^3$, $x = 4$; revolve about $y = 8$.
11. $y^2 = x^3$, $x = 4$; revolve about the y-axis.
12. $y^2 = x^3$, $y = 8$, $x = 0$; revolve about x-axis.
13. $y^2 = x^3$, $y = 8$, $x = 0$; revolve about $x = 4$.
14. $y^2 = x^3$, $y = 8$, $x = 0$; revolve about the y-axis.
15. $y^2 = x^3$, $x = 4$; revolve about $x = 4$.
16. $y^2 = x^3$, $x = 4$; revolve about the x-axis.
17. $y^2 = x^3$, $y = 8$, $x = 0$; revolve about $y = 8$.
18. The smaller area cut from the circle $x^2 + y^2 = 25$ by the line $y = 2$ is revolved about the x-axis. Find the volume of the solid which is swept out.

VOLUMES BY SLICING

In the preceding section we defined the volume of a solid of revolution as a limiting summation of the volumes of circular disks. The bases of the disks, or slices, were all perpendicular to a line. We now extend the idea of the summation of volumes of slices to find the volumes of certain solids where the slices are not circular.

Suppose that a given solid possesses parallel cross sections which are perpendicular to a line, and that the area of any cross section is expressible in terms of the distance from a point of the line. We let the line be the x-axis and let the solid extend from $x = a$ to $x = b$ (Fig. 18).



Then we partition the interval $[a, b]$ into subinterval and pass a plane through a point of each subinterval perpendicular to the x-axis. If the area of the cross section at x, expressed in

terms of x, is $A(x)$, the volume of a typical slice is $\Sigma A(x)\,\Delta x$. We indicate a summation of slices by $\Sigma A(x)\,\Delta x$ and define the volume of the solid to be the value of the integral $\int_a^b A(x)\,dx$.

Example-1 :

The base of a solid is bounded by the parabola $2y^2 = 3x$ and the line $x = 6$. Every section of the solid perpendicular to the x-axis is a square. Find the volume of the solid.

Solution :

Visualize the solid as resting on the base as shown in figure and extending outward from the base. The shaded rectangle is the base of typical square slice with length $2y$.

∴ The area of the cross section at x is

$$A(x) = (2y)^2 = 4y^2 = 2(2y^2) = 2(3x) = 6x$$

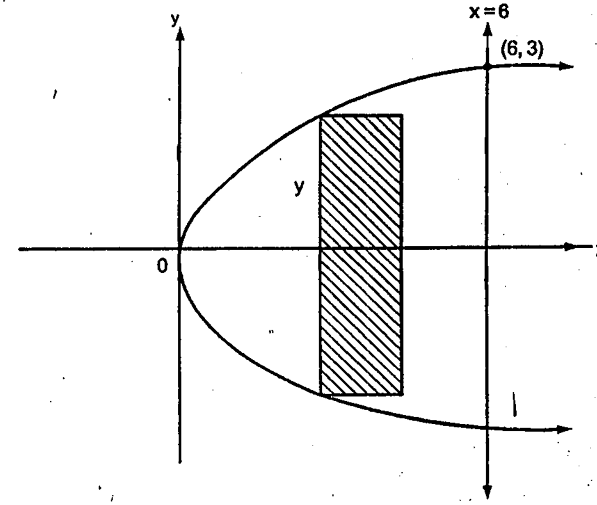
Hence volume of solid is

$$V = \int_0^a A(x)\,dx$$

$$= \int_0^6 6x\,dx$$

$$= [3x^2]_0^6$$

$$= 108$$



Example-2 :

A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.

Solution :

From figure we say that the area of cross section at x is $A(x) = x^2$

The squares lie on the planes from $x = 0$ to $x = 3$

$\therefore$ Volume of solid is

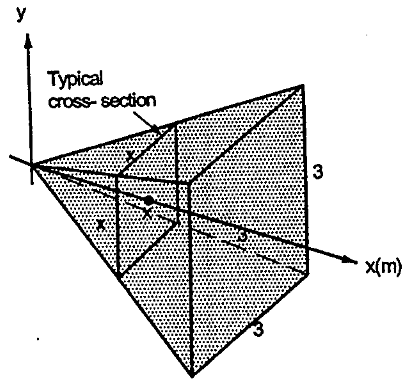
$$V = \int_0^a A(x)\,dx$$

$$= \int_0^3 x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^3$$

$$= \frac{3^3}{3}$$

$= 9$ m^3.



Example-3 :

A solid has a circular base of radius 4 units. Find the volume of the solid if every plane section perpendicular to a fixed diameter is an equilateral triangle.

Solution :

Take the circle as in the figure with x-axis as the fixed diameter. The equation of circle is $x^2 + y^2 = 16$.

The cross section ABC of the solid is an equilateral triangle of side $2y$ and area

$$A(x) = \frac{1}{2}(2y)(\sqrt{3}y)$$

$$= \sqrt{3}\,y^2$$

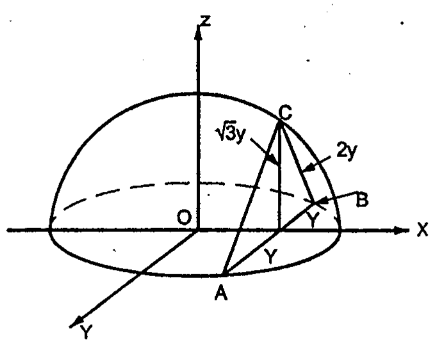
$$= \sqrt{3}\,(16 - x^2)$$

$\therefore$ Also from figure $-4 \le x \le 4$.

$$\therefore \quad V = \int_a^b A(x)\,dx = \int_{-4}^{4} \sqrt{3}(16 - x^2)\,dx$$

$$= \sqrt{3}\left[16x - \frac{x^3}{3}\right]_{-4}^{4}$$

$$= \sqrt{3}\left[\left(64 - \frac{64}{3}\right) - \left(-64 + \frac{64}{3}\right)\right]$$



$$= \sqrt{3}\left[128 - \frac{128}{3}\right]$$

$$= \sqrt{3} \times 128\left[1 - \frac{1}{3}\right]$$

$$= \sqrt{3} \times 128\left(\frac{2}{3}\right) = \frac{256\sqrt{3}}{3} = \frac{256}{\sqrt{3}}$$

EXERCISE

1. The base of a solid is a circle of radius a, and every cross section perpendicular to a diameter is a square. Find the volume of the solid.
2. A solid has a circular base of radius 4 and every plane section perpendicular to a diameter is an isosceles right triangle having its hypotenuse in the base. Find the volume of the solid.
3. The area bounded by the parabola $y^2 = 4x$ and the line $x = 4$ is the base of a solid. Every plane section perpendicular to the x-axis is an isosceles triangle of constant height 8 whose base is a chord of the parabola. Find the volume of the solid.
4. A solid is generated by a variable square which moves with its plane perpendicular to the x-axis. The ends of one side of the square are on the x-axis and the curve $xy = 4$. Find the volume of the solid generated as the square moves from $x = 1$ to $x = 4$.
5. A solid is generated by a variable equilateral triangle which moves with its plane perpendicular to the y-axis. Find the volume of the solid generated if the ends of a side of the triangle extend from the line $x = -y$ to the curve $x = \sqrt{y}$ and the triangle moves from $y = 0$ to $y = 9$.
6. Find the volume of the wedge in Example-2 by using a slab perpendicular to the x-axis as an element of volume. Observe that sections perpendicular to the x-axis are rectangles.
7. Find the volume of the wedge cut from a right circular cylinder of radius 8 in. by a plane passing through a diameter of the base at an angle of 60° with the base.
8. A pyramid is 60 ft. high, and every plane section x ft. from the base is a square of side $\frac{1}{3}(60 - x)$. Find the volume of the pyramid by integration. Check your result by using the formula for the volume of a pyramid.
9. A solid has as its base the ellipse $4x^2 + 9y^2 = 36$. Find the volume if every cross section perpendicular to the x-axis is (a) a square; (b) a semicircle; (c) an isosceles right triangle with hypotenuse in the base.

RECTIFICATION

DERIVATIVE OF AN ARC

Rectification is the process of computing the length of all arc of a curve. The curve may have different representations, like cartesian, polar and parametric. So, we shall be dealing with all the three forms. Besides, the curve could be expressed as a combination of arcs of two different curves yielding a new closed curve. In this case, the length of arc will be its perimeter. The idea of finding the length of arc is simple.

A curve is said to be **rectifiable** if it is possible to find its length.

LENGTH OF AN ARC OF A CURVE

■ **Theorem-1 :**

Let $y = f(x)$ be a cartesian representation of a curve C. Then prove that the length of arc of C between two points A and B corresponding to the x-coordinates a and b respectively is given by

$$\text{arcAB} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Proof :

Let $s(x)$ be the length of arc of curve between fixed point A on the curve and the generic point $P(x, f(x))$. Then integrating $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$ form a to b, we have,

$$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b \frac{ds}{dx}\,dx = \int_a^b ds = [s]_a^b = s(b) - s(a)$$

$$= \text{arcAB} - \text{arcAA} = \text{arcAB}$$

Hence $\text{arc AB} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

■ **Theorem-2 :**

Let $x = f(y)$ be a cartesian representation of a curve C. Then prove that the length of arc of C between two points A and B corresponding to the y-coordinates c and d respectively is given by

$$\text{arcAB} = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

■ **Theorem-3 :**

Let $r = f(\theta)$ be a polar representation of a curve C. Then prove that the length of arc of C between two points A and B corresponding to the angle $\theta = \theta_0$ and $\theta = \theta_1$ respectively, is given by

$$\text{arcAB} = \int_{\theta_0}^{\theta_1} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

Example-1 :

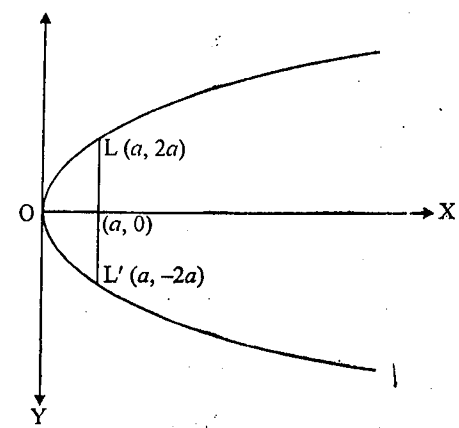
Find the length of arc of the parabola $y^2 = 4ax$, $(a > 0)$, measured from the vertex to one extremity of its latus rectum. **SPU, Sep. 2014; Nov. 2010**

Sol^n. :

We can write the given equation as $x = \frac{y^2}{4a}$. Then $\frac{dx}{dy} = \frac{y}{2a}$.

Therefore, $\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\frac{y^2}{4a^2}}$

$$= \frac{1}{2a}\sqrt{y^2+4a^2}$$



From the figure, we see that coordinates of the vertex O and top end of the latus rectum L are (0, 0) and $(a, 2a)$ respectively. Hence the required length of arc is

$$\text{arcOL} = \int_0^{2a} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$= \frac{1}{2a}\int_0^{2a} \sqrt{y^2+4a^2}\,dy$$

$$= \frac{1}{2a}\left[\frac{y}{2}\sqrt{y^2+4a^2} + \frac{4a^2}{2}\log(y+\sqrt{y^2+4a^2})\right]_0^{2a}$$

$$= \frac{1}{2a}[2\sqrt{2}a^2 + 2a^2\log(2a+2a\sqrt{2}) - 0 - 2a^2\log 2a]$$

$$= a\left[\sqrt{2} + \log\left(\frac{2a(1+\sqrt{2})}{2a}\right)\right]$$

$$= a[\sqrt{2} + \log(1+\sqrt{2})]$$

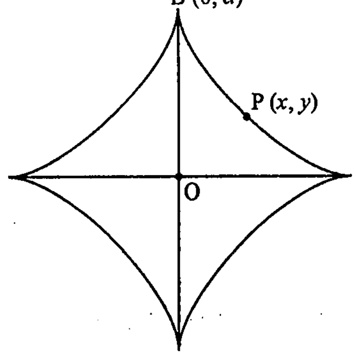
Example-2 :

(a) Find the entire length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$.

SPU, April-2015, Nov. 2013, 2011

(b) Prove that the length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ measured from $(0, a)$ to the point (x, y) is given by $\frac{3}{2}(ax^2)^{1/3}$.

Sol^n. :



Here,

$$x^{2/3} + y^{2/3} = a^{2/3}$$

$$\Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{y^{2/3}}{x^{2/3}}$$

$$= \frac{a^{2/3}}{x^{2/3}} = a^{2/3}x^{-2/3} \quad \text{... (1)}$$

From the figure, the entire length of the astroid is

$$4 \times \text{arcAB} = 4\int_a^0 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 4\int_a^0 a^{1/3}x^{-1/3}\,dx$$

$$= 4a^{1/3}\left[\frac{x^{2/3}}{2/3}\right]_a^0$$

$$= 4a^{1/3}\left(\frac{-a^{2/3}}{2/3}\right)$$

$$= -6a.$$

Since the length of an arc is always positive, we infer that the entire length of the astroid is $6a$.

(b) The required arc length $= \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^x a^{1/3}x^{-1/3}\,dx$ (by (1))

$$= a^{1/3}\left[\frac{x^{2/3}}{2/3}\right]_0^x = a^{1/3}\frac{x^{2/3}}{2/3}$$

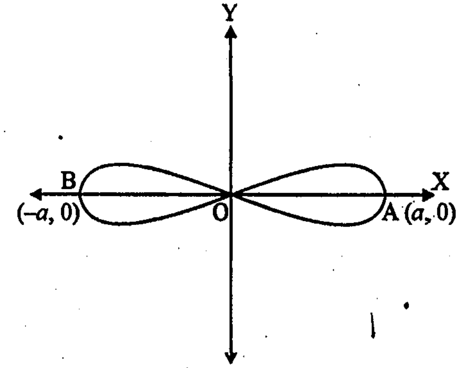
$$= \frac{3}{2}a^{1/3}x^{2/3} = \frac{3}{2}(ax^2)^{1/3}.$$

Example-3 :

Show that the entire length of the curve $x^2(a^2 - x^2) = 8a^2y^2$ is $\pi a\sqrt{2}$.

Sol^n. : **SPU, April-2016**

The given curve is symmetric about all X-axis, Y-axis and the origin. Putting $y = 0$, we get $x \in \{0, \pm a\}$. Also since $y^2 = \frac{x^2(a^2-x^2)}{8a^2}$, we have $y = \frac{x\sqrt{a^2-x^2}}{2\sqrt{2}a}$. Hence $-a \le x \le a$ is the only possibility for getting y real. The shape of the given curve is as shown in the figure. It contains two equal loops.



Here $8a^2y^2 = x^2(a^2 - x^2)$

$$\Rightarrow 16a^2y\frac{dy}{dx} = 2x(a^2-x^2) + x^2(-2x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{x(a^2-2x^2)}{8a^2y}$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{x(a^2-2x^2)}{8a^2y}\right)^2$$

$$= 1+\frac{[x(a^2-2x^2)]^2}{8a^2x^2(a^2-x^2)}$$

$$= \frac{8a^4 - 8a^2x^2 + a^4 - 4a^2x^2 + 4x^4}{8a^2(a^2-x^2)}$$

$$= \frac{(3a^2-2x^2)^2}{8a^2(a^2-x^2)}$$

From the figure, we say that the entire length

$$= 4\text{arcOA}$$

$$= 4\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 4\int_0^a \frac{3a^2 - 2x^2}{2a\sqrt{2}\sqrt{a^2 - x^2}}\,dx$$

$$= 4\int_0^a \left[\frac{a^2}{2a\sqrt{2}\sqrt{a^2 - x^2}} + 2\,\frac{a^2 - x^2}{2a\sqrt{2}\sqrt{a^2 - x^2}}\right]dx$$

$$= 4\int_0^a \left[\frac{a}{2\sqrt{2}\sqrt{a^2 - x^2}} + \frac{\sqrt{a^2 - x^2}}{a\sqrt{2}}\right]dx$$

$$= 4\left[\frac{a}{2\sqrt{2}}\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{a\sqrt{2}}\left(\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right)\right]_0^a$$

$$= 4\left[\frac{a}{\sqrt{2}}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2 - x^2}}{2\sqrt{2}a}\right]_0^a$$

$$= 4\left[\frac{a}{\sqrt{2}}\sin^{-1}(1) + 0\right] = 4\,\frac{a}{\sqrt{2}}\,\frac{\pi}{2} = \pi a\sqrt{2}.$$

Example-4 :

Find the length of the cardioid $r = a(1 + \cos\theta)$ lying outside the circle $r = -a\cos\theta$.

SPU, December-2015, April-2015

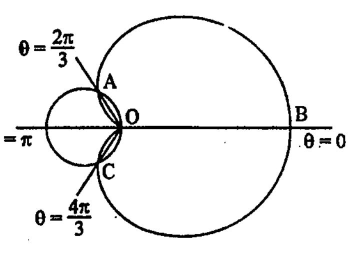
Sol^n. :

First we find the angle between two curves at the point of their intersection.

By comparing them, we get $a(1 + \cos\theta) = -a\cos\theta \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \pi \pm \frac{\pi}{3}$

$\Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$. From the figure, we see that the given curve is symmetric about the polar axis and the required arc length is arc ABC = 2arcBA.

Now for the curve $r = a(1 + \cos\theta)$.



$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2(1 + \cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2(1 + 2\cos\theta + \cos^2\theta) + a^2\sin^2\theta$$

$$= 2a^2(1 + \cos\theta)$$

$$= 4a^2\cos^2\left(\frac{\theta}{2}\right).$$

Hence the required arc length

$$2\text{arcBA} = 2\int_0^{2\pi/3}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$= 2\int_0^{2\pi/3} 2a\cos\left(\frac{\theta}{2}\right)d\theta$$

$$= 8a\left[\sin\frac{\theta}{2}\right]_0^{2\pi/3}$$

$$= 8a\left(\sin\frac{\pi}{3} - 0\right) = 4a\sqrt{3}.$$

INTRINSIC EQUATION

■ **Definition :**

Intrinsic equation : Let A be a fixed point on a curve C and P be a generic point on the curve. Let $\psi(P)$ denote the angle between the tangents to the curve at points A and P. Also, let s = arcAP. Then the relation between s and ψ is called the **intrinsic equation** of the curve.

It is customary to fix origin (or pole) as the fixed point A if it lies on the curve. Otherwise we mention the fixed point explicitly. We follow this convention throughout this section including exercise. Now we obtain the intrinsic equations of the curve represented in different forms.

■ **Theorem-1 :**

Obtain the intrinsic equations of the curve represented in (i) Cartesian form (ii) Polar form

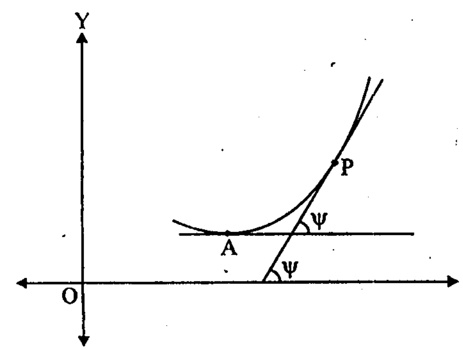
Proof :

(1) Cartesian form :

Let $A(a, b)$ be a fixed point and $P(x, y)$ be a generic point on the curve $y = f(x)$. We develop the equation in a particular case when the tangent to the curve at A is parallel to the X-axis. Then

$$s = \int_a^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \quad \ldots\ldots\ldots\ldots (1)$$

and $\tan\psi = \frac{dy}{dx}$. .... (2)



Eliminating x from (1) and (2) we get a relation

$$F(s, \psi) = 0$$

which is the intrinsic equation of the curve in cartesian form.

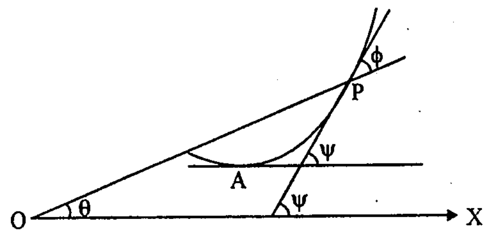
Note : If the curve is represented in the form $x = f(y)$ or in a parametric form, then the intrinsic equation can be obtained similarly by eliminating y or the parameter t respectively. However, in the polar form, the coordinates are changed, so we give equation in this case separately.

(2) Polar form :

Let $A(r_1, \theta_1)$ be a fixed point and $P(r, \theta)$. We develop the equation in a particular case when the tangent to the curve at A is parallel to the polar axis.

Then,

$$s = \int_{\theta_1}^{\theta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad \dots (3)$$



Now from the figure,

$$\psi = \theta + \varphi \qquad \dots (4)$$

where φ is the angle between the radius vector and the tangent at point P. We also know that

$$\tan\varphi = r\frac{d\theta}{dr} \qquad \dots (5)$$

Eliminating φ and θ from (3), (4) and (5) we get

$$F(s, \psi) = 0 \qquad \dots\dots\dots\dots\dots (6)$$

which is the intrinsic equation of the curve in polar form.

Example-1 :

Find the intrinsic equation of the Cardioid $r = a(1 + \cos\theta)$. Hence prove that $s^2 + 9\rho^2 = 16a^2$, where ρ is the radius of curvature at any point of the curve.

Sol^n. : **SPU, April-2016**

Here $r = a(1 + \cos\theta)$. Therefore $\frac{dr}{d\theta} = -a\sin\theta$.

$$\text{Hence } \tan\varphi = r\frac{d\theta}{dr} = -\frac{1+\cos\theta}{\sin\theta} = -\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right).$$

$$\text{Hence, } \varphi = \frac{\pi}{2} + \frac{\theta}{2}$$

$$\text{Also, } \psi = \theta + \varphi = \theta + \frac{\pi}{2} + \frac{\theta}{2} = \frac{3\theta}{2} + \frac{\pi}{2} \qquad \dots\dots\dots\dots\dots (1)$$

$$\text{Now, } s = \int_{\theta_1}^{\theta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\theta} \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\, d\theta$$

$$= a\int_0^{\theta} \sqrt{2(1+\cos\theta)}\, d\theta$$

$$= a\int_0^{\theta} \sqrt{4\cos^2\frac{\theta}{2}}\, d\theta$$

$$= 2a\int_0^{\theta} \cos\frac{\theta}{2}\, d\theta$$

$$= 2a\left[2\sin\frac{\theta}{2}\right]_0^{\theta}$$

$$= 4a\sin\frac{\theta}{2}$$

$$= 4a\sin\left(\frac{\psi}{3} - \frac{\pi}{6}\right), \text{ (by (1))} \qquad \dots\dots\dots\dots\dots (2)$$

which is the required intrinsic equation. By differentiating (2), we have

$$\rho = \frac{ds}{d\psi} = \frac{4a}{3}\cos\left(\frac{\psi}{3} - \frac{\pi}{6}\right).$$

Hence $3\rho = 4a\cos\left(\frac{\psi}{3} - \frac{\pi}{6}\right)$. So, $s^2 + 9\rho^2 = 16a^2$.

Example-2 :

Show that the intrinsic equation of the curve $y^3 = ax^2$ is $27s = 8a(\sec^3\psi - 1)$.

SPU, December-2015, November-2013

Sol^n. :

We can write the given equation as $x = \frac{1}{\sqrt{a}}y^{3/2}$. Then $\frac{dx}{dy} = \frac{3}{2}\sqrt{\frac{y}{a}}$.

Here the tangent to the curve at the origin is Y-axis. Therefore, $\tan\psi = \frac{dx}{dy}$. That is,

$$\tan\psi = \frac{3}{2}\sqrt{\frac{y}{a}} \qquad \dots\dots\dots\dots\dots (1)$$

Now, $s = \int_0^y \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$

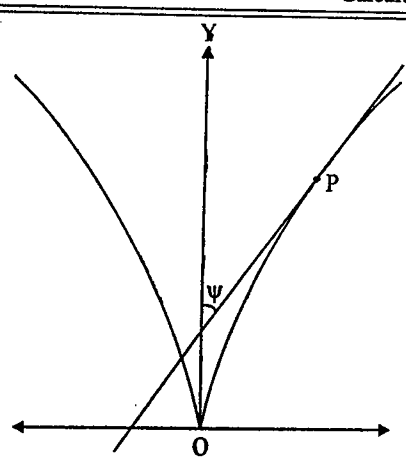
$$= \int_0^y \sqrt{1+\frac{9y}{4a}}\, dy$$

$$= \frac{4a}{9}\frac{2}{3}\left[\left(1+\frac{9y}{4a}\right)^{3/2}\right]_0^y$$

$$\Rightarrow \quad 27s = 8a\left[\left(1+\frac{9y}{4a}\right)^{3/2} - 1\right]$$

$$\Rightarrow \quad 27s = 8a\,[(1+\tan^2\psi)^{3/2} - 1] \quad \text{(by (1))}$$

$$\Rightarrow \quad 27s = 8a(\sec^3\psi - 1).$$



Example-3 :

Determine the value of θ at the point of intersection of $r = a(1 - \cos\theta)$ and $r = a\cos\theta$.

SPU, April-2015

Sol^n. :

Compare $r = a(1 - \cos\theta)$ and $r = a\cos\theta$

we get $a\,(1 - \cos\theta) = a\cos\theta$

$\Rightarrow \quad a - a\cos\theta = a\cos\theta$

$\Rightarrow \quad a = 2a\cos\theta$

$\Rightarrow \quad \cos\theta = \frac{1}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{3}, -\frac{\pi}{3}$

$\Rightarrow \quad \boxed{\theta = \pm\frac{\pi}{3}}$

Example-4 :

Find the length of curve $y = \cos hx$ measured from (0, 1) to (1, e). **SPU, April-2015**

Sol^n. :

Length of curve $= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad \left[\begin{array}{l} y = \cos hx \\ \therefore\ \frac{dy}{dx} = \sin hx \end{array}\right]$

$$= \int_0^1 \sqrt{1+\sin h^2 x}\, dx$$

$$= \int_0^1 \cos hx\, dx$$

$$= [\sin hx]_0^1$$

$$= \sin h1 - \sin h0$$

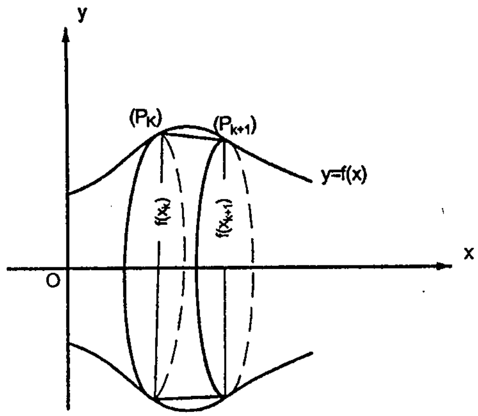
$$= \sin h1$$

EXERCISE

1. Find the length of arc of the parabola $x^2 = 4ay$, $(a > 0)$, cut off by its latus rectum.
2. Find the length of the curve $y = \log\left(\frac{e^x - 1}{e^x + 1}\right)$ from $x = 1$ to $x = 2$.
3. Find the length of arc of $x = a\sin 2t(1 + \cos 2t)$, $y = a\cos 2t(1 - \cos 2t)$ measured from th origin to any point.
4. Prove that the length of the arc of $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$ measured fron $t = 0$ to any point is $\frac{at^2}{2}$.
5. Find the length of the arc of the curve $x = e^t\sin t$, $y = e^t\cos t$ from $t = 0$ to $t = \frac{\pi}{2}$.
6. Show that the length of the arc of the semi-cubical parabola $ay^2 = x^3$ from the verte to the point (a, a) is $\frac{a}{27}(13^{3/2} - 8)$.
7. Find the perimeter of the curve $r = a(1 + \cos\theta)$ and show that the arc in the upper hal of the curve is bisected by $\theta = \frac{\pi}{3}$.
8. Prove that the line $4r\cos\theta = 3a$ divides the cardioid $r = a(1 + \cos\theta)$ into two parts such the length of the arc on either side of the line are equal.
9. Show that the arc of the upper half of the cardioid $r = a(1 - \cos\theta)$ is bisected by $\theta = \frac{2\pi}{3}$.
10. Find the intrinsic equation of the cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$ and prove tha $x^2 + S^2 = 16a^2$.
11. Find the intrinsic equation of the parabola $y^2 = 4ax$.
12. Find the intrinsic equation of the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$ when s being measured from (i) $\theta = \frac{\pi}{4}$, (ii) $\theta = 0$.
13. Find the intrinsic equation of $3ay^2 = 2x^3$.

AREA OF A SURFACE OF REVOLUTION

Consider Fig. If the arc and broken line extending from A to B are revolved about the x-axis, each generates a surface. The segments of the broken line trace n frustums of right circular cones. We shall say that the sum of the areas of the frustums approximate the area of the surface traced by the arc $\widehat{AB}$, and we shall define the area traced by the arc as the limit of the area traced by the broken as n increases and the length of each segment approaches zero.



In figure we show an enlargement of a representative segment of the broken line AB and the subtended arc. In a complete revolution the chord $P_k P_{k+1}$ generates a frustum of a cone whose slant height is the length of the chord and whose bases have $f(x_k)$ and $f(x_{k+1})$ are radii. According to a formula from solid geometry, the area of a frustum of a right circular cone is

$$\pi(r_1 + r_2)\, l,$$

where r_1 and r_2 are the radii of the bases and l is the slant height. Hence the area S_k traced by $P_k P_{k+1}$ is

$$S_k = \pi[f(x_k) + f(x_{k+1})]\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

By the law of the mean,

$$\Delta y_k = f'(x_k')\,\Delta x_k, \quad \text{where } x_k < x_k' < x_{k+1}$$

And

$$f(x_k) + f(x_{k+1}) = 2f(x_k''), \quad \text{where } x_k'' = \frac{1}{2}(x_k + x_{k+1})$$

Hence $$S_k = 2\pi f(x_k'')\sqrt{1 + [f'(x_k')]^2}\,\Delta x_k,$$

and the area traced by the broken line AB is

$$S_n = 2\pi \sum_{k=1}^{n} f(x_k'')\sqrt{1 + [f'(x_k')]^2}\,\Delta x_k$$

We define the area of the surface of revolution of the arc $\widehat{AB}$ as the limit of S_n as $n \to \infty$ and each $\Delta x_k \to 0$. Thus, by the Bliss theorem

$$S = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x_k')]^2}\,dx \qquad \text{... (1)}$$

Formula (1) can be expressed more generally by

$$S = 2\pi \int y\, ds, \qquad \text{... (2)}$$

where $$ds = \sqrt{(dx)^2 + (dy)^2}$$

Definition :

Surface Area for Revolution about the x-axis

If the function $y = f(x) \geq 0$ is continuously differentiable on $[a, b]$, the area of the surface generated by revolving the curve $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx \qquad \text{... (3)}$$

Surface Area for Revolution about the y-axis

If $x = g(x) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the curve $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

$$= \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\,dy \qquad \text{... (4)}$$

Surface Area for Revolution for Parametrized Carves:

If a smooth curve $x = f(t)$, $y = g(t)$, $a \leq t \leq b$, is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. Revolution about the x-axis ($y \geq 0$) :

$$S = \int_a^b 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \qquad \text{... (5)}$$

2. Revolution about the y-axis ($x \geq 0$) :

$$S = \int_a^b 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \qquad \text{... (5)}$$

Example-1 :

The arc of the curve $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$ is revolved about x-axis. Find the area of the surface thus generated.

Solution :

Here $y = \cos x$

$$\therefore \quad \frac{dy}{dx} = -\sin x$$

$$\therefore \quad \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\sin^2 x}$$

$\therefore$ Area of surface of revolution

$$S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi \int_a^{\pi/2} \cos x\sqrt{1+\sin^2 x}\,dx$$

Let $\sin x = t$ then $\cos x\,dx = dt$

Also For $x = 0$, $t = 0$ and For $x = \frac{\pi}{2}$, $t = 1$

$$\therefore \quad S = 2\pi \int_0^1 \sqrt{1+t^2}\,dt$$

$$= 2\pi\left[\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln\left(1+\sqrt{1+t^2}\right)\right]_0^1$$

$$= \pi\left[\sqrt{2} + \ln(1+\sqrt{2}) - \ln 2\right]$$

$$= \pi\left[\sqrt{2} + \ln\left(\frac{1+\sqrt{2}}{2}\right)\right]$$

Example-2 :

Find the area of the surface generated by revolving about the y-axis the arc of the parabola $x = 2\sqrt{y}$ from $y = 0$ to $y = 3$.

Solution :

Here $x = 2\sqrt{y}$

$$\therefore \quad \frac{dx}{dy} = \frac{1}{\sqrt{y}}$$

$$\text{Also} \quad \sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\frac{1}{y}} = \frac{1}{\sqrt{y}}\sqrt{y+1}$$

$$\therefore \quad S = 2\pi \int_c^d x\sqrt{1+\frac{dx}{dy}}\,dy$$

$$= 2\pi \int_0^3 2\sqrt{y}\cdot\frac{1}{\sqrt{y}}\sqrt{y+1}\,dy$$

$$= 4\pi\left[\frac{(y+1)^{3/2}}{3/2}\right]_0^3$$

$$= \frac{8\pi}{2}[4^{3/2} - 1]$$

$$= \frac{8\pi}{2}[8-1]$$

$$= \frac{56\pi}{3}$$

Example-3 :

Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \le x \le 2$ about x-axis.

Solution :

$$y = 2\sqrt{x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{x}}$$

$$\therefore \quad \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{x}} = \frac{1}{\sqrt{x}}\sqrt{x+1}$$

$$\therefore \quad S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi \int_1^2 2\sqrt{x}\,\frac{1}{\sqrt{x}}\sqrt{x+1}\,dx$$

$$= 4\pi\left[\frac{(x+1)^{3/2}}{3/2}\right]_1^2$$

$$= \frac{8\pi}{3}[3^{3/2} - 2^{3/2}]$$

Example-4 :

The line $x = 1 - y$, $0 \le y \le 1$, is revolved about the y-axis to generate the cone. Find its lateral surface area.

Solution :

Here $x = 1 - y$ $\quad \therefore \; \frac{dx}{dy} = -1$

$$\therefore \quad S = 2\pi \int_c^d x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= 2\pi \int_0^1 (1 - y)\sqrt{1+1}\, dy$$

$$= 2\sqrt{2}\pi \left[y - \frac{y^2}{2}\right]_0^1 = 2\sqrt{2}\pi\left[1 - \frac{1}{2}\right] = \sqrt{2}\pi.$$

Example-5 :

Find the area of the surface swept out by revolving the circle $x^2 + y^2 = 1$, $y > 0$ about x-axis.

Solution :

We know that the parametric equation of $x^2 + y^2 = 1$ is

$x = \cos t$, $y = \sin t$, $0 \le t \le \pi$

$$\therefore \quad \frac{dx}{dt} = -\sin t, \; \frac{dy}{dt} = \cos t$$

$$\therefore \quad S = 2\pi \int_a^b y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 2\pi \int_a^\pi \sin t \sqrt{\cos^2 t + \sin^2 t}\, dt$$

$$= 2\pi[-\cos t]_0^\pi = 2\pi\,[1 + 1] = 4\pi.$$

Example-6 :

Find the area of the surface generated by revolving the curve $y = x^3$, $0 \le x \le \frac{1}{2}$, about x-axis.

Solution :

Here $y = x^3$ $\quad \therefore \; \frac{dy}{dx} = 3x^2$

$$\therefore \quad \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + 9x^4}$$

$$\therefore \quad S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_0^{1/2} x^3 \sqrt{1 + 9x^4}\, dx$$

$$= \frac{2\pi}{36} \int_0^{1/2} (36x^3)(1 + 9x^4)^{1/2}\, dx$$

$$= \frac{\pi}{18}\left[\frac{(1 + 9x^4)^{3/2}}{3/2}\right]_0^{1/2}$$

$$= \frac{\pi}{27}\left[\left(1 + \frac{9}{16}\right)^{3/2} - 1\right]$$

$$= \frac{\pi}{27}\left[\left(\frac{25}{16}\right)^{3/2} - 1\right]$$

$$= \frac{\pi}{27}\left[\left(\frac{5}{4}\right)^3 - 1\right] = \frac{\pi}{27}\left[\frac{125}{64} - 1\right] = \frac{61\pi}{1728}$$

EXERCISE

Find the area of the surface generated by revolving the arc in each of Problems 1–11 about the x-axis.

1. $y = x$ from $x = 0$ to $x = 2$
2. $y = \frac{4}{3}x + 2$ from $x = 0$ to $x = 3$
3. $y = x^3$ from $x = 0$ to $x = 2$
4. $y = \sqrt{4 - x^2}$ from $x = -2$ to $x = 2$
5. $y = \sin x$ from $x = 0$ to $x = \pi$

6. $3y = \sqrt{36 - 4x^2}$ from $x = -3$ to $x = 3$
7. $y = e^x$ from $x = -2$ to $x = 2$
8. $x = t^2, y = t$ from $t = 1$ to $t = 3$
9. $x = \ln t^2, y = t^2$ from $t = 1$ to $t = 3$
10. $x = a\cos\theta, y = a\sin\theta$ from $\theta = 0$ to $\theta = \pi$
11. $x = a\sin^3 t, y = a\cos^3 t$ from $t = 0$ to $t = \frac{1}{2}\pi$

Find the area of the surface generated by revolving the arc in each of Problems 12–16 about the y-axis.

12. $y = x$ from $x = 0$ to $x = 2$
13. $x = \cos y$ from $y = 0$ to $y = \frac{1}{2}\pi$
14. $y^2 - x^2 = a^2$ from $y = a$ to $y = 3a$
15. $y = \ln(x^2 - 1)$ from $x = 2$ to $x = 4$
16. $x = 2t^3, y = 3t^2$ from $t = 0$ to $t = 2$
17. The cardioid $r = 1 + \cos\theta$ is revolved about the polar axis. Find the area of the surface generated.
18. The arc of the spiral $r = e^\theta$ from $\theta = 0$ to $\theta = \pi$ is revolved about the polar axis. Find the area of the surface generated.
19. The upper half of the circle $r = 2a\cos\theta$ is revolved about the polar axis. Find the area of the surface generated.
20. Find the area of the surface generated when on arch of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ is revolved about the x-axis.
21. The ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is revolved about the x-axis. Find the area of the prolate spheroid thus generated. What would be the area of the oblate spheroid generated by revolving the ellipse about the y-axis ?
22. A circle of radius a is revolved about a line in the plane of the circle at a distance $b > a$ from the center of the circle. Find the area of the surface of the torus thus generated.

Finding Surface Area:

23. Find the lateral (side) surface area of the cone generated by revolving the line segment $y = \frac{x}{2}$, $0 \le x \le 4$, about the x-axis. Check your answer with the geometry formula.

    Lateral surface area = $\frac{1}{2}$ × base circumference × slant height.

24. Find the lateral surface area of the cone generated by revolving the line segment $y = \frac{x}{2}$, $0 \le x \le 4$, about the y-axis. Check your answer with the geometry formula.

    Lateral surface area = $\frac{1}{2}$ × base circumference × slant height.

25. Find the lateral surface area of the cone frustum generated by revolving the line segment $y = \frac{x}{2} + \frac{1}{2}$, $1 \le x \le 3$, about the x-axis. Check your result with the geometry formula

    Frustum surface area = $\pi(r_1 + r_2)$ × slant height.

26. Find the lateral surface area of the cone frustum generated by revolving the line segment $y = \frac{x}{2} + \frac{1}{2}$, $1 \le x \le 3$, about the y-axis. Check your result with the geometry formula

    Frustum surface area = $\pi(r_1 + r_2)$ × slant height.

Find the area of the surfaces generated by revolving the curves in Exercise 27–36 about the indicated axes. If you have a grapher, you may want to graph these curve to see what they look like.

27. $y = x$, $0 \le x \le 1$; x-axis
28. $y = \sqrt{x}$, $\frac{3}{4} \le x \le \frac{15}{4}$; x-axis
29. $y = \sqrt{2x - x^2}$, $0.5 \le x \le 1.5$; x-axis
30. $y = \sqrt{x + 1}$, $1 \le x \le 5$; x-axis

(**Hint** : Express $ds = \sqrt{dx^2 + dy^2}$ in terms of dy, and evaluate the integral $S = \int 2\pi y \, ds$ with appropriate limits.)

31. $x = \frac{y^3}{3}$, $0 \le y \le 1$; y-axis
32. $x = \left(\frac{1}{3}\right)y^{3/2} - y^{1/2}$, $1 \le y \le 3$; y-axis
33. $x = \left(\frac{y^4}{4}\right) + \frac{1}{(8y^2)}$, $1 \le y \le 2$; x-axis
34. $y = \left(\frac{1}{3}\right) \cdot (x^2 + 2)^{3/2}$, $0 \le x \le \sqrt{2}$; y-axis

(**Hint :** Express $ds = \sqrt{dx^2 + dy^2}$ in terms of dx, and evaluate the integral $S = \int 2\pi x \, ds$ with appropriate limits.)

MULTIPLE CHOICE QUESTIONS

■ **Fill in the Blanks :**

1. **If** $J_n = \int_0^{\pi/2} \sin^n x \, dx$ **then** J_n = ________s

   (a) $\frac{n+1}{n} J_{n-2}$ (b) $\frac{n-1}{n} J_{n-2}$ (c) $\frac{n}{n-1} J_{n-2}$ (d) None

2. $\int_0^{\pi/2} \sin^{10} x \, dx$ = ________.

   (a) $\frac{63}{256}$ (b) $\frac{63}{512}$ (c) $\frac{63\pi}{512}$ (d) None

3. If $J_{p,q} = \int_0^{\pi/2} \sin^p x \cos^q x \, dx$, then $J_{p,q} =$ ______.

(a) $\frac{p+1}{p+q} J_{p-2,q}$ (b) $\frac{p-1}{p+q} J_{p+2,q}$ (c) $\frac{p}{p-q} J_{p,q}$ (d) $\frac{p-1}{p+q} J_{p-2,q}$

4. Surface area for revolution about the x-axis is S = ______.

(a) $\int_a^b 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$ (b) $\int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

(c) $\int_a^b \pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$ (d) $\int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)}\, dx$

5. If $I_n = \int \sin^n x \, dx$ then $I_n = -\frac{\sin^{n-1} x \cos x}{n} +$ ______.

(a) $\frac{n-1}{n} I_{n-2}$ (b) $\frac{n-1}{n} I_{n-1}$ (c) $\frac{n+1}{n} I_{n-1}$ (d) $\frac{n+1}{n} I_{n-2}$

6. If $I_n = \int \tan^n x \, dx$ then $I_n = \frac{\tan^{n-1} x}{n-1} -$ ______.

(a) I_{n-1} (b) I_{n+1} (c) I_{n+2} (d) I_{n-2}

7. If $J_n = \int_0^{\pi/4} \tan^n x \, dx$ then $J_n =$ ______.

(a) $\frac{1}{n-1} - J_{n+2}$ (b) $\frac{1}{n-1} - J_{n-2}$ (c) $\frac{1}{n+1} - J_{n-2}$ (d) $\frac{1}{n-1} + J_{n-2}$

8. If the area bounded by $x = f(y)$, $y = c$, $y = d$ and y-axis is revolved about the y-axis then the volume of the solid thus generated is V = ______.

(a) $\int_c^d \pi x^2 dy$ (b) $2\pi \int_c^d x^2 dy$ (c) $\int_c^d \pi y^2 dy$ (d) $\int_c^d 2\pi y^2 dy$

9. Volume by the Washer method is V = ______.

(a) $\int_a^b (y_2^2 - y_1^2)\, dx$ (b) $\pi \int_a^b (y_2 - y_1)\, dx$ (c) $\pi \int_a^b (y_2^2 + y_1^2)\, dx$ (d) $\pi \int_a^b (y_2^2 - y_1^2)\, dx$

10. Volume by the cylindrical shell method is V = ______.

(a) $2\pi \int_a^b xy \, dx$ (b) $\pi \int_a^b xy \, dx$ (c) $\pi \int_a^b x^2 \, dx$ (d) None

ANSWERS

1. (b), 2. (c), 3. (d), 4. (b), 5. (a), 6. (d), 7. (b), 8. (a), 9. (d), 10. (a).

■ ✳ ■

UNIT 4 Curvature, Partial Derivatives, Vector Functions

CURVATURE

Let $f : I \to R$ be a sufficiently many times differentiable function on an interval I. Then the points on the graph of $y = f(x)$ is curve. However, not all curves could be represented as a graph of such a real valued function on intervals viz, the figure of a circle with centre (0, 0) and radius 1 in the XY-plane R^2 is one such example of a curve. In this situation, we have to represent the equation of the circle as $x = \cos t$; $y = \sin t$, $t \in [0, 2\pi]$. These are called the **parametric equations of the circle.** Also, let us think of a spring put in R^3. Then the points of this spring is a curve. Thus formally we have the following definition of a curve.

■ **Definition :**

Locus of the curve : Let I be a closed interval and $x = x(t)$, $y = y(t)$ and $z = z(t)$ be real-valued differentiable functions defined on I. Then the points $(x(t), y(t), z(t))$ in the space is called a **locus of the curve** represented by the parametric equations $x = x(t)$, $y = y(t)$ and $z = z(t)$, $t \in$ I.

Throughout this chapter we shall be concerned only with curves lying in the XY-plane. For such curves we have $z = 0$. Hence they are described by $x = x(t)$ and $y = y(t)$.

Planer curve : A curve lying only in one plane is called a planer curve.

■ **Definition :**

(1) Cartesian representation of the curve : Let $x = x(t)$, $y = y(t)$ be a curve. If we eliminate t and obtain a relation $g(x, y) = 0$, then this form is called the Cartesian representation of the curve.

(2) Cartesian equation of the curve : If $g(x, y) = 0$ can be written in the form $y = f(x)$ [respectively, $x = f(y)$], then $y = f(x)$ [respectively, $x = f(y)$] is called the Cartesian equation of the curve.

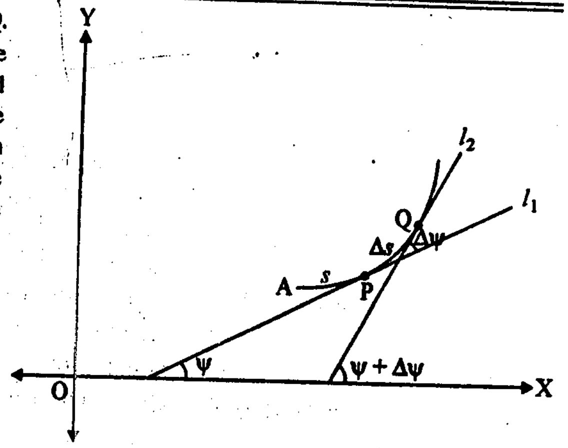
■ **Example :**

Let I = [0, 1] and $x = t$, $y = t^2$. Then this is a curve that can also be represented by the cartesian equation $y = x^2$.

■ **Definition :**

Curvature : Let $y = f(x)$ be a curve. Fix a point A on this curve. For a point P on the curve, let s = arc AP be the arc length from A to P. For a point Q on the curve, let

$s + \Delta s$ = arc AQ so that Δs = arc PQ. Let l_1, l_2 be the tangents to the curve at the points P and Q making angles ψ and $\psi + \Delta\psi$ respectively, with a fixed line in the plane. Clearly the angle between these two tangents is $\Delta\psi$, called the **total bending** or **total curvature** of the arc between P and Q. Hence the **average bending** or the **average curvature** of the curve between these two points relative to the arc length is given by $\frac{\Delta\psi}{\Delta s}$. The **bending** or the **curvature** of the curve at P is defined to be $\frac{d\psi}{ds} = \lim_{Q \to P} \frac{\Delta\psi}{\Delta s} = \lim_{\Delta s \to 0} \frac{\Delta\psi}{\Delta s}$.



DERIVATIVE OF AN ARC

■ **Proposition-1 :**

Fix a point $A(x_0, y_0)$ on a curve given by $y = f(x)$. For a point $P(x, f(x))$ on the curve, let s be the arc length of arc AP, (Clearly, s is a function of x). Then prove that

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

SPU, April-2015, 2016, September-2014

Proof :

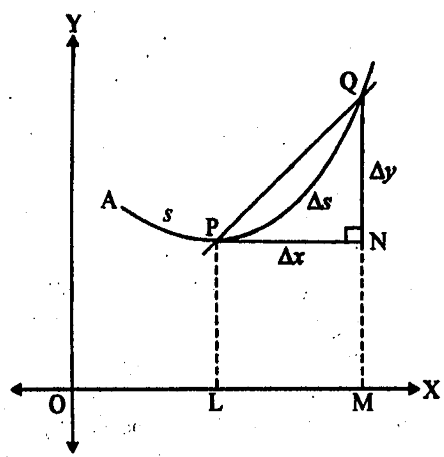
Let $y = f(x)$ represent the given curve and A be a fixed point on it. Let $P(x, y)$ be a generic point on the curve. Let the arc AP = s. Take a point $Q(x + \Delta x, y + \Delta y)$ on the curve near to P. Let arc AQ = $s + \Delta s$. From the right angled triangle ΔPNQ, we have,

$$PQ^2 = PN^2 + NQ^2 = (\Delta x)^2 + (\Delta y)^2$$

$$\Rightarrow \left(\frac{PQ}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$

$$\Rightarrow \left(\frac{PQ}{\Delta s}\right)^2 \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$

$$\Rightarrow \left(\frac{\text{chord PQ}}{\text{arc PQ}}\right)^2 \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$



Taking Q → P, we get chord PQ → arc PQ and $\Delta x \to 0$. Hence,

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \Rightarrow \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

The proof of the following corollary is left to the reader.

■ **Corollary-1 :**

Let $x = x(t)$ and $y = y(t)$ be the parametric equations of a curve. Then prove that

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

■ **Exercise :**

In the proposition 1, suppose that the curve is represented by $x = f(y)$. Then deduce that the derivative of the arc length $\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$.

■ **Definition :**

Polar representation of the curve : Let $h(x, y) = 0$ be a cartesian representation of a curve. By substituting $x = r\cos\theta$ and $y = r\sin\theta$, in this form we get a representation $g(r, \theta) = 0$ of the curve called a **polar representation of the curve.**

We shall be mainly dealing with the form $r = f(\theta)$ of the curve.

■ **Theorem-1 :**

For a polar equation $r = f(\theta)$ of a curve, prove that

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

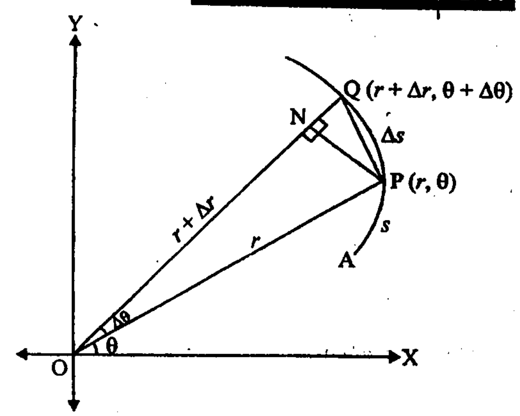
SPU, November-2013, 2011

Proof :

Let $r = f(\theta)$ represent the given curve and A be a fixed point on it.

Let P (r, θ) be a generic point on the curve. Let arc AP = s. Take a point $Q(r + \Delta r, \theta + \Delta\theta)$ on the curve near to P. Let AQ = $s + \Delta s$.

From the right angled triangle ΔONP as shown in figure, we have,



$$\sin\Delta\theta = \frac{PN}{OP} = \frac{PN}{r} \Rightarrow PN = r\sin\Delta\theta$$

and $$\cos\Delta\theta = \frac{ON}{OP} = \frac{ON}{r} \Rightarrow ON = r\cos\Delta\theta$$

Also, from the figure,

$$\begin{aligned} NQ &= OQ - ON \\ &= r + \Delta r - r\cos\Delta\theta \\ &= r(1 - \cos\Delta\theta) + \Delta r \\ &= 2r\sin^2\frac{\Delta\theta}{2} + \Delta r \end{aligned}$$

Now from the right angled traingle ΔPNQ, we have,

$$PQ^2 = PN^2 + NQ^2$$

$$\Rightarrow PQ^2 = r^2\sin^2\Delta\theta + \left(2r\sin^2\frac{\Delta\theta}{2} + \Delta r\right)^2$$

$$\Rightarrow \left(\frac{PQ}{\Delta\theta}\right)^2 = r^2\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)^2 + \left[r\sin\left(\frac{\Delta\theta}{2}\right)\left(\frac{\sin\left(\frac{\Delta\theta}{2}\right)}{\frac{\Delta\theta}{2}}\right) + \frac{\Delta r}{\Delta\theta}\right]^2$$

$$\Rightarrow \left(\frac{\text{cord } PQ}{\text{arc } PQ}\right)^2\left(\frac{\text{arc } PQ}{\Delta\theta}\right)^2 = r^2\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)^2 + \left[r\sin\left(\frac{\Delta\theta}{2}\right)\left(\frac{\sin\left(\frac{\Delta\theta}{2}\right)}{\frac{\Delta\theta}{2}}\right) + \frac{\Delta r}{\Delta\theta}\right]^2$$

Taking $Q \to P$, we get chord $PQ \to$ arc PQ and $\Delta\theta \to 0$. Hence,

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 \Rightarrow \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

■ **Example-1 :**

For the curve $r^n = a^n\cos n\theta$, prove that $\frac{ds}{d\theta} = a(\sec n\theta)^{\frac{n-1}{n}}$.

Soln. :

Here $r^n = a^n\cos n\theta$. Taking log on both sides,

$$n\log r = n\log a + \log(\cos n\theta)$$

By differentiating this we get,

$$\frac{n}{r}\frac{dr}{d\theta} = -n\frac{\sin n\theta}{\cos n\theta} \Rightarrow r_1 = \frac{dr}{d\theta} = -r\tan n\theta$$

$$\Rightarrow r^2 + r_1^2 = r^2(1 + \tan^2 n\theta) = r^2\sec^2 n\theta$$

$$\Rightarrow \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = r\sec n\theta = a(\cos n\theta)^{\frac{1}{n}}\sec n\theta = a(\sec n\theta)^{\frac{n-1}{n}}$$

RADIUS OF CURVATURE

■ **Definition :**

Radius of the curvature : Let P be a point on a curve such that the curvature of the curve at P is nonzero. Then the radius of the curvature at P is defined to be the reciprocal of the curvature at P and is denoted by ϱ. That is, $\varrho = \frac{ds}{d\psi}$.

■ **Theorem-2 :**

Let $y = f(x)$ be a curve and P be a point on it. Then prove that the radius of curvature at P is given by

$$\varrho = \frac{(1 + y_1^2)^{\frac{3}{2}}}{y_2}$$

where $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$.

SPU, April-2016, Sep. 2014, Nov. 2011, 2010

Proof :

Let $y = f(x)$ be the given curve. Then $\tan\psi = \frac{dy}{dx}$. Differentiating with respect to s, we get,

$$\sec^2\psi\frac{d\psi}{ds} = \frac{d}{ds}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{ds} = y_2\frac{dx}{ds}$$

$$\Rightarrow (1 + \tan^2\psi)\frac{d\psi}{ds} = y_2\frac{dx}{ds}$$

$$\Rightarrow (1 + y_1^2)\frac{d\psi}{ds} = y_2\frac{dx}{ds}$$

$$\Rightarrow \varrho = \frac{ds}{d\psi} = \frac{(1 + y_1^2)}{y_2}\frac{ds}{dx}$$

$$\Rightarrow \varrho = \frac{(1 + y_1^2)}{y_2}\sqrt{1 + y_1^2}$$

$$\Rightarrow \varrho = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

■ Theorem-3 :

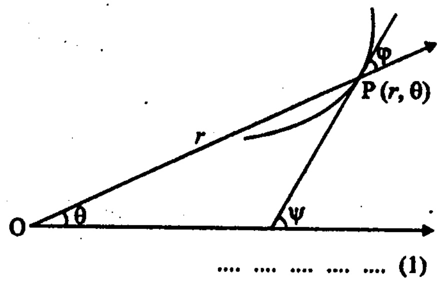
Let $r = f(\theta)$ be a polar form of a curve with a point P on it. Then prove that the radius of curvature at P is given by

$$\varrho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

where $r_1 = f'(\theta)$ and $r_2 = f''(\theta)$. **SPU, Dec. 2015, April-2015; Nov. 2010**

Proof :

From the figure it is clear that $\psi = \theta + \varphi$.



Hence, $\dfrac{d\psi}{ds} = \dfrac{d\theta}{ds} + \dfrac{d\varphi}{ds}$

$$= \frac{d\theta}{ds} + \frac{d\varphi}{d\theta}\frac{d\theta}{ds}$$

$$= \frac{d\theta}{ds}\left(1 + \frac{d\varphi}{d\theta}\right) \quad \text{.... .... (1)}$$

We know that $\tan\varphi = \dfrac{r}{r_1}$. Differentiating this with respect to θ, we get,

$$\sec^2\varphi\left(\frac{d\varphi}{d\theta}\right) = \frac{r_1^2 - rr_2}{r_1^2}$$

$$\Rightarrow \quad \frac{d\varphi}{d\theta} = \left(\frac{r_1^2 - rr_2}{r_1^2}\right)\left(\frac{1}{1 + \tan^2\varphi}\right)$$

$$= \left(\frac{r_1^2 - rr_2}{r_1^2}\right)\frac{1}{\left(1 + \dfrac{r^2}{r_1^2}\right)} = \frac{r_1^2 - rr_2}{r_1^2 + r^2}$$

We also know that $\dfrac{ds}{d\theta} = \sqrt{r^2 + r_1^2}$. Hence by (1), we get,

$$\frac{d\psi}{ds} = \frac{1}{\sqrt{r^2 + r_1^2}}\left(1 + \frac{r_1^2 - rr_2}{r_1^2 + r^2}\right)$$

$$= \frac{r^2 + 2r_1^2 - rr_2}{(r^2 + r_1^2)^{3/2}}$$

Hence, $\varrho = \dfrac{ds}{d\psi} = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$

■ Example-2 :

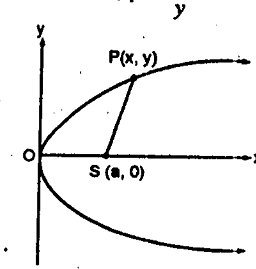
Prove that if ϱ is the radius of curvature at any point P on the parabola $y^2 = 4ax$ and S is its focus, then prove that $\varrho^2 \propto SP^3$.

Soln. :

Let P (x, y) be any point on the given parabola. If the coordinates of the focus S is given by (a, 0), then

$$SP = \sqrt{(x-a)^2 + y^2} = \sqrt{x^2 - 2ax + a^2 + 4ax} = x + a.$$

Now we find ϱ for the given parabola $y^2 = 4ax$. Here $2yy_1 = 4a$. That is, $y_1 = \dfrac{2a}{y}$. Also, $y_2 = -\dfrac{2a}{y^2}y_1 = -\dfrac{4a^2}{y^3}$.



Hence, $\varrho = \dfrac{(1 + y_1^2)^{3/2}}{y_2} = \dfrac{\left(1 + \dfrac{4a^2}{y^2}\right)^{3/2}}{\dfrac{-4a^2}{y^3}} = -\dfrac{(y^2 + 4a^2)^{3/2}}{4a^2}$

$$\Rightarrow \quad \varrho^2 = \frac{(4ax + 4a^2)^3}{16a^4} = \frac{64a^3(x+a)^3}{16a^4} = \frac{4(x+a)^3}{a} = \frac{4}{a}SP^3.$$

This proves that $\varrho^2 \propto SP^3$.

■ Example-3 :

Show that the radius of curvature at any point of the curve $x = ae^\theta(\cos\theta - \sin\theta)$, $y = ae^\theta(\sin\theta + \cos\theta)$ is twice the perpendicular distance of the tangent at the point form the origin. **SPU, September-2014**

Soln. :

Here, $\dfrac{dx}{d\theta} = ae^\theta(\cos\theta - \sin\theta) + ae^\theta(-\sin\theta - \cos\theta) = -2ae^\theta\sin\theta$

Similarly, $\dfrac{dy}{d\theta} = 2ae^\theta\cos\theta$

Hence, $y_1 = \dfrac{dy}{dx} = -\cot\theta$ and $y_2 = \operatorname{cosec}^2\theta\dfrac{d\theta}{dx} = \dfrac{\operatorname{cosec}^3\theta}{-2ae^\theta}$.

Thus, $\varrho = \dfrac{(1 + y_1^2)^{3/2}}{y_2} = \dfrac{(1 + \cot^2\theta)^{3/2}}{-\left(\dfrac{\operatorname{cosec}^3\theta}{2ae^\theta}\right)} = -2ae^\theta \approx 2ae^\theta$

Now the equation of the tangent at a point is

$$y - ae^\theta(\sin\theta + \cos\theta) = \frac{dy}{dx}[x - ae^\theta(\cos\theta - \sin\theta)]$$

$$\Rightarrow \quad y - ae^\theta(\sin\theta + \cos\theta) = -\cot\theta[x - ae^\theta(\cos\theta - \sin\theta)]$$

$$\Rightarrow \quad [y - ae^\theta(\sin\theta + \cos\theta)]\sin\theta = -\cos\theta[x - ae^\theta(\cos\theta - \sin\theta)]$$

$\Rightarrow \quad y\sin\theta - ae^{\theta}\sin^2\theta - ae^{\theta}\cos\theta\sin\theta + x\cos\theta - ae^{\theta}\cos^2\theta + ae^{\theta}\cos\theta\sin\theta = 0$

$\Rightarrow \quad y\sin\theta + x\cos\theta - ae^{\theta} = 0$

Hence the length of the perpendicular distance of the tangent from the origin is

$$p = \left|\frac{-ae^{\theta}}{\cos^2\theta + \sin^2\theta}\right| = ae^{\theta}. \text{ Hence } \rho = 2p.$$

■ **Example-4 :**

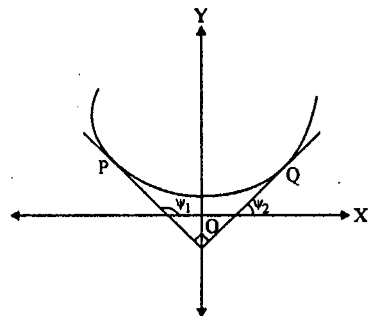
For the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ prove that $\rho = 4a\cos\left(\frac{\theta}{2}\right)$. Also show that $\rho_1^2 + \rho_2^2 = 16a^2$, where ρ_1, ρ_2 are the radius of curvature at the points where the tangents are perpendicular.

SPU, Nov. 2011, 2010

Sol^n. :

$$\frac{dx}{d\theta} = a(1+\cos\theta), \quad \frac{dy}{d\theta} = a\sin\theta.$$

Therefore,



$$y_1 = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\cos^2\frac{\theta}{2}} = \tan\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \quad y_2 = \frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)\frac{d\theta}{dx} = \left(\frac{1}{2\cos^2\left(\frac{\theta}{2}\right)}\right)\left(\frac{1}{2a\cos^2\left(\frac{\theta}{2}\right)}\right) = \frac{1}{4a\cos^4\left(\frac{\theta}{2}\right)}$$

$$\text{Hence, } \rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \left(1+\tan^2\left(\frac{\theta}{2}\right)\right)^{3/2} 4a\cos^4\left(\frac{\theta}{2}\right) = 4a\sec^3\left(\frac{\theta}{2}\right)\cos^4\left(\frac{\theta}{2}\right) = 4a\cos\left(\frac{\theta}{2}\right)$$

If $P(\theta_1)$ and $Q(\theta_2)$ are the points at which the tangents are perpendicular, then $\rho_1 = 4a\cos\left(\frac{\theta_1}{2}\right)$ and $\rho_2 = 4a\cos\left(\frac{\theta_2}{2}\right)$. If the tangents at these points make the angles ψ_1 and ψ_2 with the X-axis respectively, then $\tan\psi_1 = \frac{dy}{dx} = \tan\left(\frac{\theta_1}{2}\right)$. Therefore, $\psi_1 = \frac{\theta_1}{2}$. But $\psi_1 - \psi_2 = \frac{\pi}{2}$. Therefore, $\frac{\theta_1}{2} + \frac{\theta_2}{2} = \frac{\pi}{2}$.

$$\therefore \quad \frac{\theta_2}{2} = \frac{\pi}{2} - \frac{\theta_1}{2}$$

$$\text{Hence, } \rho_1^2 + \rho_2^2 = 16a^2\left[\cos^2\left(\frac{\theta_1}{2}\right) + \cos^2\left(\frac{\pi}{2} - \frac{\theta_1}{2}\right)\right] = 16a^2\left[\cos^2\left(\frac{\theta_1}{2}\right) + \sin^2\left(\frac{\theta_1}{2}\right)\right] = 16a^2$$

■ **Example-5 :**

For the curve $r = a(1 - \cos\theta)$, prove that $\rho^2 \propto r$. Also prove that if ρ_1 and ρ_2 are radii of the curvature at the ends of a chord through the pole, $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$.

Sol^n. :

SPU, December-2015

Here, $r_1 = a\sin\theta$ and $r_2 = a\cos\theta$.

$$\text{Hence, } \rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

$$= \frac{(a^2(1-\cos\theta)^2 + a^2\sin^2\theta)^{3/2}}{a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a^2(1-\cos\theta)\cos\theta}$$

$$= \frac{[a^2(1-2\cos\theta+\cos^2\theta+\sin^2\theta)]^{3/2}}{a^2(1-2\cos\theta+\cos^2\theta+2\sin^2\theta-\cos\theta+\cos^2\theta)}$$

$$= \frac{[2a^2(1-\cos\theta)]^{3/2}}{3a^2(1-\cos\theta)}$$

$$= \frac{\left(4a^2\sin^2\frac{\theta}{2}\right)^{3/2}}{6a^2\sin^2\frac{\theta}{2}} = \frac{4}{3}a\sin\frac{\theta}{2}$$

Thus,

$$\rho^2 = \frac{16}{9}a^2\sin^2\frac{\theta}{2} \qquad \ldots\ldots\ldots\ldots (1)$$

$$= \frac{8}{9}a^2(1-\cos\theta) = \frac{8ar}{9}$$

$\Rightarrow \quad \rho^2 \propto r.$

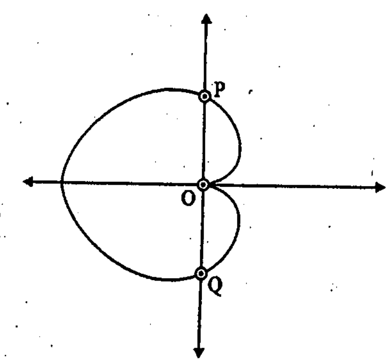
Let $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$ be the ends of the chord through the pole. Then $\theta_2 - \theta_1 = \pi$. Then $\rho_1^2 = \frac{16}{9}a^2\sin^2\frac{\theta_1}{2}$, $\rho_2^2 = \frac{16}{9}a^2\sin^2\frac{\theta_2}{2}$.

$$\text{Hence, } \rho_1^2 + \rho_2^2 = \frac{16}{9}a^2\left(\sin^2\frac{\theta_1}{2} + \sin^2\frac{\theta_2}{2}\right)$$

$$= \frac{16}{9}a^2\left[\sin^2\frac{\theta_1}{2} + \sin^2\left(\frac{\pi+\theta_1}{2}\right)\right] \quad (\because\ \theta_2 - \theta_1 = \pi)$$

$$= \frac{16}{9}a^2\left(\sin^2\frac{\theta_1}{2} + \cos^2\frac{\theta_1}{2}\right)$$

$$= \frac{16}{9}a^2.$$



- **Example-6 :**

For the curve $y = a\sin 2x$ find $\frac{ds}{dx}$.

Sol^n^. :

$y = a\sin 2x \quad \therefore \quad \frac{dy}{dx} = 2a\cos 2x$

We know that $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+4a^2\cos^2 2x}$

- **Example-7 :**

Find radius of curvature of any point on the curve $s = 8a\sin^2\left(\frac{\psi}{6}\right)$.

SPU, November-2013

Sol^n^. :

$$\varrho = \frac{ds}{d\psi} = 8a\cdot 2\sin\left(\frac{\psi}{6}\right)\cdot\cos\left(\frac{\psi}{6}\right)\cdot\frac{1}{6}$$

$$= \frac{4}{3}a\sin 2\left(\frac{\psi}{6}\right) = \frac{4}{3}a\sin\left(\frac{\psi}{3}\right)$$

- **Example-8 :**

Find ϱ for $r = a(1+\cos\theta)$. SPU, November-2013

Sol^n^. :

$\therefore \quad r_1 = -a\sin\theta,\ r_2 = -a\cos\theta$

We know that

$$\varrho = \frac{[r^2+r_1^2]^{3/2}}{r^2+2r_1^2-rr_2}$$

$$= \frac{[a^2(1+\cos\theta)^2+a^2\sin^2\theta]^{3/2}}{a^2(1+\cos\theta)^2+2a^2\sin^2\theta+a(1+\cos\theta)\,a\cos\theta}$$

$$= \frac{[a^2+2a^2\cos\theta+a^2\cos^2\theta+a^2\sin^2\theta]^{3/2}}{(1+\cos\theta)\,[a^2(1+\cos\theta)+2a^2(1-\cos\theta)+a^2\cos\theta]}$$

$$= \frac{[2a^2+2a^2\cos\theta]^{3/2}}{(1+\cos\theta)\,[a^2+a^2\cos\theta+2a^2-2a^2\cos\theta+a^2\cos\theta]}$$

$$= \frac{[2a^2(1+\cos\theta)]^{3/2}}{(1+\cos\theta)\,3a^2}$$

$$= \frac{\left[2a^2\cdot 2\cos^2\left(\frac{\theta}{2}\right)\right]^{3/2}}{3a^2\cdot 2\cos^2\left(\frac{\theta}{2}\right)}$$

$$= \frac{\left[4a^2\cos^2\left(\frac{\theta}{2}\right)\right]^{3/2}}{6a^2\cos^2\left(\frac{\theta}{2}\right)} = \frac{8a^3\cos^3\left(\frac{\theta}{2}\right)}{6a^2\cos^2\left(\frac{\theta}{2}\right)} = \frac{4}{3}a\cos\left(\frac{\theta}{2}\right)$$

MULTIPLE CHOICE QUESTIONS

- **Fill in the Blanks :**

1. If $r^n = a^n\cos n\theta$ then r_1 = ______.

(a) $-r\tan n\theta$ (b) $r\tan n\theta$ (c) $\tan n\theta$ (d) $-r\cot n\theta$

2. If $y^2 = 4ax$, S is its focus and ϱ is radius of curvature at point P, then ϱ^2 = ______.

(a) $\frac{4}{a}(SP)^2$ (b) $\frac{4}{a}(SP)^3$ (c) $4(SP)^3$ (d) $(SP)^3$

3. If $y^2 = 4ax$, S is its focus and ϱ is radius of curvature at point P, then $\varrho^2 \propto$ ______.

(a) SP (b) $\frac{4}{a}(SP)^3$ (c) $(SP)^3$ (d) $(SP)^2$

4. If $x = ae^\theta(\cos\theta-\sin\theta)$; $y = ae^\theta(\cos\theta+\sin\theta)$ then y_1 = ______.

(a) $\tan\theta$ (b) $\cot\theta$ (c) $-\tan\theta$ (d) $-\cot\theta$

5. If $x = a(\theta+\sin\theta)$; $y = a(1-\cos\theta)$ then y_1 = ______.

(a) $\tan\left(\frac{\theta}{2}\right)$ (b) $\frac{\tan\theta}{2}$ (c) $\cot\left(\frac{\theta}{2}\right)$ (d) $-\tan\left(\frac{\theta}{2}\right)$

6. If $r = a(1 - \cos\theta)$ then r_2 = ______.

(a) $a\sin\theta$ (b) $a\cos\theta$ (c) $-a\cos\theta$ (d) $-a\sin\theta$

7. If $r = a(1 - \cos\theta)$ then ρ^2 = ______.

(a) $\frac{ar}{9}$ (b) $\frac{8r}{9}$ (c) $\frac{8ar}{9}$ (d) r

8. If $r = a(1 - \cos\theta)$ then $\rho^2 \propto$ ______.

(a) $\frac{ar}{9}$ (b) r^2 (c) $\frac{8ar}{9}$ (d) r

9. If $y^2 = 4ax$, $a > 0$, then $1+\left(\frac{dx}{dy}\right)^2$ = ______.

(a) $\frac{y^2+4a^2}{4a^2}$ (b) $\frac{y^2-4a^2}{4a^2}$ (c) $\frac{y^2}{4a^2}$ (d) y^2

10. If $x^{2/3} + y^{2/3} = a^{2/3}$ then y_1 = ______.

(a) $\left(\frac{y}{x}\right)^{1/3}$ (b) $-\left(\frac{y}{x}\right)^{1/3}$ (c) $-\left(\frac{y}{x}\right)^{2/3}$ (d) $\left(-\frac{y}{x}\right)^{1/3}$

11. If $x^{2/3} + y^{2/3} = a^{2/3}$ then $1 + y_1^2$ = ______.

(a) $\left(\frac{a}{x}\right)^{1/3}$ (b) $\left(\frac{a}{x}\right)^{4/3}$ (c) $\left(\frac{a}{x}\right)^{2/3}$ (d) $\frac{a}{x}$

12. If $x^{2/3} + y^{2/3} = a^{2/3}$ then $\sqrt{1+y_1^2}$ = ______.

(a) $\left(\frac{a}{x}\right)^{2/3}$ (b) $\frac{a}{x}$ (c) $\left(\frac{a}{x}\right)^{3/2}$ (d) $\left(\frac{a}{x}\right)^{1/3}$

13. $x^2(a^2 - x^2) = 8a^2y^2$ is symmetric about ______.

(a) x–axis, y–axis and origin all (b) x–axis
(c) y–axis (d) origin

14. For $x^2(a^2 - x^2) = 8a^2y^2$, $x \in$ ______.

(a) $(-a, a)$ (b) $[-a, a]$ (c) $[0, a]$ (d) $[-a, 0]$

15. For $x^2(a^2 - x^2) = 8a^2y^2$, x-intercepts are x = ______.

(a) $\pm a$ (b) a, 0 (c) $\pm a$, 0 (d) $\pm a$, ± 1

16. For $x^2(a^2 - x^2) = 8a^2y^2$, y-intercepts are y = ______.

(a) a (b) $-a$ (c) $\pm a$ (d) 0

17. The angle between $r = a(1 + \cos\theta)$ and $r = -a\cos\theta$ at the point of intersection is θ = ______.

(a) $\pi \pm \frac{\pi}{3}$ (b) $\pi + \frac{\pi}{3}$ (c) $\pi - \frac{\pi}{3}$ (d) $\pm\frac{\pi}{3}$

18. The curvature of the curve at a point is ______.

(a) Rate of change of the arc length with respect to the independent variable
(b) Rate of change of the arc length with respect to the angle made by the tangents.
(c) Rate of the change of the bending of the tangent with respect to the arc length
(d) None of the above

19. For the curve $y = f(x)$ the derivative of the arc length s, with respect to x, is given by ______.

(a) $\sqrt{1+\left(\frac{dx}{dy}\right)^2}$ (b) $\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$ (c) $\sqrt{1+\left(\frac{dy}{dt}\right)^2}$ (d) None of the above

20. Curvature of the line $2x + 3y = 1$ is ____.

(a) 0
(b) $\sqrt{1+p^2}$, where p is the perpendicular distance from the origin
(c) 1
(d) None of the above

21. Curvature of the circle $x^2 + y^2 = 1$ is ______.

(a) 0 (b) 2 (c) 1 (d) None of the above

22. The reciprocal of the curvature at a points is known as ______.

(a) Radius of the curve at a point (b) Radius of curvature at a point
(c) Rate of bending of the tangent (d) None of the above

23. For the curve $y = f(x)$ the radius of curvature at a given point is given by ______.

(a) $\frac{(1+y_1^2)^{3/2}}{y_2}$ (b) $\frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$ (c) $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ (d) None of the above

24. ______ has infinite radius of curvature at any point

(a) Circle with radius 4
(b) Parabola $y^2 = 4ax$
(c) Line $y = x$
(d) None of the above

25. The radius of the curvature of the curve $s = a\psi$ is ______.

(a) $\frac{(1+a^2)}{a}$
(b) a
(c) 0
(d) None of the above

26. The intrinsic equation is ______.

(a) a function of arc length
(b) a relation between arc length(s) and the angle made by the tangent(ψ)
(c) a function F $(s, x, y) = 0$
(d) None of the above

27. At a point on a curve, with non zero curvature, the radius of curvature and the curvature are ______.

(a) Additive inverse of each other
(b) Multiplicative inverse of each other
(c) Equal
(d) None

28. For $r = f(\theta)$ ______ is not true.

(a) $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$
(b) $\tan\phi = \frac{r}{r_1}$
(c) $\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$
(d) $\frac{ds}{d\theta} = \sqrt{1 + \left(\frac{dr}{d\theta}\right)^2}$

29. Intrinsic equation of a curve involves ______.

(a) cartesian coordinates only
(b) Polar coordinates only
(c) Parametric coordinates only
(d) None of these

30. For a curve $y = f(x)$ the radius of curvature at a point (x, y) is given by ______.

(a) $\frac{(1+y_1^2)^{3/2}}{y_2}$
(b) $\frac{(1+y_1^2)^{3/2}}{y_1}$
(c) $\frac{(1+y_1^2)^{3/2}}{y_2^2}$
(d) $\frac{(y_1^2 + y_2^2)}{y_2}$

31. Rectification is a process of ______.

(a) Measuring the length of arc on a curve
(b) Finding the curvature at a point on the curve
(c) Finding the radius of curvature
(d) None of these

ANSWERS

1. (a),	2. (b),	3. (c),	4. (d),	5. (a),	6. (b),	7. (c),	8. (d),
9. (a),	10. (b),	11. (c),	12. (d),	13. (a),	14. (b),	15. (c),	16. (d),
17. (a),	18. (c),	19. (c),	20. (a),	21. (a),	22. (b),	23. (a),	24. (c),
25. (b),	26. (b),	27. (b),	28. (d),	29. (d),	30. (a),	31. (a).	

SHORT QUESTIONS

1. Show that curvature of a circle is constant and is equal to the reciprocal of its radius.
2. Show that curvature of a straight line is zero.
3. Find $\frac{ds}{dx}$ for the following curves.

(a) $y = a\cosh\left(\frac{x}{a}\right)$
(b) $y = a\log\left(\frac{a^2}{a^2 - x^2}\right)$

4. Find $\frac{ds}{dx}$ for the following curves.

(a) $x = a(t - \sin)$; $y = a(1 - \cos t)$
(c) $x = ae^t \sin t$; $y = ae^t \cos t$
(b) $x = a(\cos t + t\sin t)$; $y = a(\sin t - t\cos t)$

5. Find $\frac{ds}{d\theta}$ for the following curves.

(a) $r = a(1 - \cos\theta)$
(b) $r^2 = a^2\cos 2\theta$

6. Find the radius of curvature at any point of the following curve.

(a) $s = a\psi$
(b) $s = a\log(\sec\psi)$
(c) $s = a\sec^3\psi$
(d) $s = 8a\sin^2\frac{\psi}{6}$
(e) $s = 8\tan\psi$
(f) $s = a\log\left(\tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right)\right)$

EXERCISE

1. Find the radius of curvature at any point on the curve.

(a) $y = a\cosh\frac{x}{a}$ (b) $xy = a^2$

(c) $x^{2/3} + y^{2/3} = a^{2/3}$ (d) $y = a\log\left(\sec\frac{x}{a}\right)$

(e) $y^2 = 4ax$ (f) $x = a\cos^3 t$; $y = a\sin^3 t$

(g) $x = \frac{a\cos t}{t}$; $y = \frac{a\sin t}{t}$ (h) $r = a\theta$

(i) $r = \frac{a}{\theta}$ (j) $r = a(1 - \cos\theta)$

(k) $r(1 + \cos\theta) = a$ (l) $r = 2(\cos\theta + \sin\theta)$

(m) $r^n = a^n \sin n\theta$ (n) $r = ae^{\theta\cot\alpha}$

(o) $r = a\sec^2\frac{\theta}{2}$ (p) $\theta = \frac{\sqrt{r^2 - a^2}}{a} - \cos^{-1}\frac{a}{r}$

(q) $x = a\sin 2t(1 + \cos 2t)$; $y = a\cos 2t(1 - \cos 2t)$

(r) $x = a(t + \sin t)$; $y = a(1 - \sin t)$

(s) $x = a(\cos t + t\sin t)$; $y = a(\sin t - t\cos t)$

2. Find the radius of curvature at the given point.

(a) $y = 4\sin x - \sin 2x$, at $x = \frac{\pi}{2}$.

(b) $x^2 y = a(x^2 + a^2)$, at $(a, 2a)$

(c) $x^3 + y^3 = 3axy$, at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

(d) $\sqrt{x} + \sqrt{y} = 1$, at the point where it meets the line $y = x$.

(e) $x = 3 + 4\cos t$; $y = 4 + 3\sin t$, at (3, 7)

(f) $x = a(1 + \sin\theta)\cos\theta$; $y = a(1 + \cos\theta)\sin\theta$, at $\theta = -\frac{\pi}{4}$

(g) $r = a\sin n\theta$, at the pole.

(h) $r = a\cos n\theta$, at $r = a$.

3. Show that the radius of curvature at any point of $x = a\cos^3\theta$; $y = a\sin^3\theta$ is equal to three times the length of the perpendicular from the origin to the tangent.

4. Prove that the radius of curvature at the point $(-2a, 2a)$ of the curve $x^2 y = a(x^2 + y^2)$ is $-2a$.

5. Show that the ratio of the radii of the curvature of the curves $xy = a^2$ and $x^3 = 3a^2 y$ at the points which have the same abscissae varies as the square root of the ratio of the ordinates.

6. Show that the least value of $|\rho|$ for $y = \log x$ is $\frac{3\sqrt{3}}{2}$.

7. Find the points on the parabola $y^2 = 8x$ at which the radius of curvature is $7\frac{13}{16}$.

8. Find ρ for $r^m = a^m \cos m\theta$ at any point $P(r, \theta)$. Also choosing a proper value of m show that in the parabola $\rho^2 \propto SP^3$, where S is the focus.

PARTIAL DERIVATIVES

The concept of partial derivative plays a vital role in differential calculus. The different ways of limit discussed in the previous section, yield different type of partial derivatives of a function.

■ **Definitions :**

Partial derivative of function : Consider a real valued function $z = f(x, y)$ defined on $E \subset R^2$ such that E contains a neighbourhood of $(a, b) \in R^2$. Let Δa be a change in a. If the limit,

$$\lim_{\Delta a \to 0} \frac{f(a + \Delta a, b) - f(a, b)}{\Delta a}$$

exists, then it is called the *partial derivative of f with respect to x at (a, b)* and is denoted by $\left.\frac{\partial f}{\partial x}\right|_{(a,b)}$ of $f_x(a, b)$.

Similarly, let Δb be a change in b. If the limit,

$$\lim_{\Delta b \to 0} \frac{f(a, b + \Delta b) - f(a, b)}{\Delta b}$$

exists, then it is called the *partial derivative of f with respect to y at (a, b)* and is denoted by $\left.\frac{\partial f}{\partial y}\right|_{(a,b)}$ of $f_y(a, b)$ or $z_y(a, b)$.

■ **Notations :**

If the partial derivatives f_x and f_y exist at each point of E, then they are also the real valued functions on E. Further, we can obtain the partial derivatives of these functions, if they are differentiable. In these cases, we fix up the following notations.

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \text{ and } f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

The notations of derivatives of order greater than two should be clear from the above pattern.

EXERCISE

1. Find the radius of curvature at any point on the curve.

(a) $y = a\cosh\frac{x}{a}$ (b) $xy = a^2$

(c) $x^{2/3} + y^{2/3} = a^{2/3}$ (d) $y = a\log\left(\sec\frac{x}{a}\right)$

(e) $y^2 = 4ax$ (f) $x = a\cos^3 t;\ y = a\sin^3 t$

(g) $x = \frac{a\cos t}{t};\ y = \frac{a\sin t}{t}$ (h) $r = a\theta$

(i) $r = \frac{a}{\theta}$ (j) $r = a(1 - \cos\theta)$

(k) $r(1 + \cos\theta) = a$ (l) $r = 2(\cos\theta + \sin\theta)$

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(o) $r = a\sec^2\frac{\theta}{2}$ (p) $\theta = \frac{\sqrt{r^2 - a^2}}{a} - \cos^{-1}\frac{a}{r}$

(q) $x = a\sin 2t(1 + \cos 2t);\ y = a\cos 2t(1 - \cos 2t)$

(r) $x = a(t + \sin t);\ y = a(1 - \sin t)$

(s) $x = a(\cos t + t\sin t);\ y = a(\sin t - t\cos t)$

2. Find the radius of curvature at the given point.

(a) $y = 4\sin x - \sin 2x$, at $x = \frac{\pi}{2}$.

(b) $x^2 y = a(x^2 + a^2)$, at $(a, 2a)$

(c) $x^3 + y^3 = 3axy$, at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

(d) $\sqrt{x} + \sqrt{y} = 1$, at the point where it meets the line $y = x$.

(e) $x = 3 + 4\cos t;\ y = 4 + 3\sin t$, at $(3, 7)$

(f) $x = a(1 + \sin\theta)\cos\theta;\ y = a(1 + \cos\theta)\sin\theta$, at $\theta = -\frac{\pi}{4}$

(g) $r = a\sin n\theta$, at the pole.

(h) $r = a\cos n\theta$, at $r = a$.

3. Show that the radius of curvature at any point of $x = a\cos^3\theta;\ y = a\sin^3\theta$ is equal to three times the length of the perpendicular from the origin to the tangent.

4. Prove that the radius of curvature at the point $(-2a, 2a)$ of the curve $x^2 y = a(x^2 + y^2)$ is $-2a$.

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exists, then it is called the *partial derivative of f with respect to x at* (a, b) and is denoted by $\frac{\partial f}{\partial x}\Big|_{(a, b)}$ of $f_x(a, b)$.

Similarly, let Δb be a change in b. If the limit,

$$\lim_{\Delta b \to 0} \frac{f(a, b + \Delta b) - f(a, b)}{\Delta b}$$

exists, then it is called the *partial derivative of f with respect to y at* (a, b) and is denoted by $\frac{\partial f}{\partial y}\Big|_{(a, b)}$ of $f_y(a, b)$ or $z_y(a, b)$.

■ **Notations :**

If the partial derivatives f_x and f_y exist at each point of E, then they are also the real valued functions on E. Further, we can obtain the partial derivatives of these functions, if they are differentiable. In these cases, we fix up the following notations.

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \quad \text{and} \quad f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

The notations of derivatives of order greater than two should be clear from the above pattern.

Remark :

As we have seen in the above example, in general, f_{xy} and f_{yx} need not be equal, even f they exist. The following proposition gives a sufficient condition fro them to be equal. We ccept it without proof. However, we shall be dealing only with the functions f for which these wo are equal.

Proposition :

Consider a real valued function $z = f(x, y)$ defined on $E \subset R^2$ such that E contains a eighbourhood of $(a, b) \in R^2$. If f_{xy} and f_{yx} exist and are continuous, then $f_{xy} = f_{yx}$.

Throughout this chapter our blanket assumption will be that the operation of taking partial lerivation is commutative. That is, for function f of two variables $f_{xy} = f_{yx}$.

Example :

For $u = x^3 - 3xy^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Also prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Sol^n. :

Here $u = x^3 - 3xy^2$.

Hence, $\frac{\partial u}{\partial x} = 3x^2 - 3y^2$; $\frac{\partial u}{\partial y} = -6xy$; $\frac{\partial^2 u}{\partial x \partial y} = -6y = \frac{\partial^2 u}{\partial y \partial x}$

$\frac{\partial^2 u}{\partial x^2} = 6x$; $\frac{\partial^2 u}{\partial y^2} = -6x$

Hence, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$ and $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

HOMOGENEOUS FUNCTIONS

Let us observe the following expressions carefully.

(1) $f_1(x, y) = x^2y^4 - x^3y^3 + xy^5$.

(2) $f_2(x, y) = x^4y^4 - x^5y^3 + x^6y^2$.

The combined degree of x and y in each term of the first expression is 6 and that in the econd expression is 8. Can we determine whether the combined degree of x and y in each erm of the expression $\frac{x}{x^4 + y^4}$ is same or not ? It seems difficult to determine. Let us develop he following tests.

Test-1 :

Let us take $t = \frac{y}{x}$.

Then $x^2y^4 - x^3y^3 + xy^5 = x^6(t^4 - t^3 + t^5) = x^6 f(t)$

and $x^4y^4 - x^5y^3 + x^6y^2 = x^8(t^4 - t^3 + t^2) = x^8 g(t)$,

where f and g are functions of one variable t.

Test-2 :

Now, let us replace x by tx and y by ty.

Then $f_1(tx, ty) = (tx)^2(ty)^4 - (tx)^3(ty)^3 + (tx)(ty)^5 = t^6 f_1(x, y)$

and $f_2(tx, ty) = (tx)^4(ty)^4 - (tx)^5(ty)^3 + (tx)^6(ty)^2 = t^8 f_2(x, y)$.

Definitations :

Homogeneous function : A function $z = f(x, y)$ is said to be a ***homogeneous function of degree r***, if $f(tx, ty) = t^r f(x, y)$ for some real number r or if $f(x, y) = x^r g\left(\frac{y}{x}\right)$ Otherwise, f is said to be a ***non-homogeneous function***.

Example-1 :

Let $f : R^2 \setminus \{(x, y) : y = -x\} \to R$ defined by $f(x, y) = \frac{x - y}{x + y}$. Then prove that f is a homogeneous function of degree 0 and f_x and f_y exist at each point of the domain.

Sol^n. :

Clearly $f(tx, ty) = f(x, y) = t^0 f(x, y)$. Thus f is a homogeneous function of degree 0.

Now for any $(x, y) \in R^2$ with $x + y \neq 0$, we have,

$$f_x(x, y) = \frac{(x + y)(1) - (x - y)(1)}{(x + y)^2} = \frac{2y}{(x + y)^2}$$

and $$f_y(x, y) = \frac{(x + y)(-1) - (x - y)(1)}{(x + y)^2} = \frac{-2x}{(x + y)^2}.$$

Example-2 :

$f : R^2 \setminus \{(0, 0)\} \to R$ defined by $f(x, y) = \frac{\sqrt[5]{x} - \sqrt[5]{y}}{x^3 + y^3}$ is a homogeneous function of degree $-\frac{14}{5}$.

$$f(tx, ty) = \frac{(tx)^{1/5} - (ty)^{1/5}}{(tx)^3 + (ty)^3} = \frac{t^{1/5}[x^{1/5} - y^{1/5}]}{t^3[x^3 + y^3]} = t^{\frac{1}{5} - 3} f(x, y) = t^{-\frac{14}{5}} f(x, y)$$

Thus $f(x, y)$ is homogneous function of degree $-\frac{14}{5}$.

Theorem-1 : State and prove Euler's Theorem for $z = f(x, y)$

SPU, April-2015, 2016, December-2015, November-2013

Statement :

Let $z = f(x, y)$ be a real valued function defined on $E \subset R^2$. Suppose that f is a homogeneous function of degree n. If f_x and f_y exist on E, then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \qquad \dots (1)$$

Proof :

Since $z = f(x, y)$ is a homogeneous function of x, y of degree n, we can write

$$z = f(x, y) = x^n g\left(\frac{y}{x}\right) \qquad \dots (2)$$

Differentiating (2) partially with respect to x, we get,

$$\frac{\partial z}{\partial x} = nx^{n-1} g\left(\frac{y}{x}\right) + x^n g'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

Hence, $$x\frac{\partial z}{\partial x} = nx^n g\left(\frac{y}{x}\right) - x^{n-1} yg'\left(\frac{y}{x}\right) \qquad \dots (3)$$

Similarly, differentiating (2) partially with respect to y, we get,

$$\frac{\partial z}{\partial y} = x^n g'\left(\frac{y}{x}\right)\frac{1}{x} = x^{n-1} g'\left(\frac{y}{x}\right)$$

Hence, $$y\frac{\partial z}{\partial y} = yx^{n-1} g'\left(\frac{y}{x}\right) \qquad \dots (4)$$

Adding (3) and (4) we get,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n g\left(\frac{y}{x}\right) = nz$$

This completes the proof.

We note that the converse of Euler's Theorem also holds. That is, if a function $z = f(x, y)$ satisfies (1), on a certain domain, then it must be homogeneous on that domain.

Remark :

Now onwards we shall not mention the domain of the functions under discussion. Also, whenever we use the derivatives of functions under discussion, we assume them to be sufficiently many time differentiable.

Corollary-1 :

Let $z = f(x, y)$ be a real valued function defined on $E \subset R^2$. Suppose that f is a homogeneous function of degree n and that all the second order partial derivatives of f exist and are continuous.

Then prove that

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

SPU, November-2011

Proof :

Since $z = f(x, y)$ is a homogeneous function of x, y of degree n, by Euler's Theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \qquad \dots (5)$$

Differentiating (5) partially with respect to x, we have,

$$x\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial x\partial y} = n\frac{\partial z}{\partial x}$$

which, on multiplication by x, gives

$$x^2\frac{\partial^2 z}{\partial x^2} + x\frac{\partial z}{\partial x} + xy\frac{\partial^2 z}{\partial x\partial y} = nx\frac{\partial z}{\partial x}$$

Hence, $$x^2\frac{\partial^2 z}{\partial x^2} + xy\frac{\partial^2 z}{\partial x\partial y} = (n-1)x\frac{\partial z}{\partial x}$$

Similarly, differentiating (5) partially with respect to y and then multiplying the result by y, we get,

$$y^2\frac{\partial^2 z}{\partial y^2} + xy\frac{\partial^2 z}{\partial y\partial x} = (n-1)y\frac{\partial z}{\partial y} \qquad \dots (6)$$

Since $\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial^2 z}{\partial y\partial x}$, we get,

$$y^2\frac{\partial^2 z}{\partial y^2} + xy\frac{\partial^2 z}{\partial x\partial y} = (n-1)y\frac{\partial z}{\partial y}. \qquad \dots (7)$$

By adding (6) and (7) we have,

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) = n(n-1)z,$$

This completes the proof.

Corollary-2 :

Let $u = u(x, y)$ be a nonhomogeneous real valued function defined on $E \subset R^2$ and $z = \varphi(u)$ be homogeneous function of degree n. Then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{\varphi(u)}{\varphi'(u)}$$

provided $\varphi'(u) \neq 0$ for any $(x, y) \in E$.

roof :

Since $z = \varphi(u)$ is a homogeneous function of x, y of degree n, by Euler's Theorem we ave,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz = n\varphi(u)$$

$$\Rightarrow \quad x\left(\varphi'(u)\frac{\partial u}{\partial x}\right) + y\left(\varphi'(u)\frac{\partial u}{\partial y}\right) = n\varphi(u)$$

$$\Rightarrow \quad x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = n\frac{\varphi(u)}{\varphi'(u)}$$

Corollary-3 :

Let $u = u(x, y)$ be a nonhomogeneous real valued function defined on $E \subset R^2$ and $= \varphi(u)$ be homogeneous function of degree n. Then prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \psi(u)\,[\psi'(u) - 1]$$

where $\psi(u) = n\dfrac{\varphi(u)}{\varphi'(u)}$, provided $\varphi'(u) \neq 0$ for any $(x, y) \in E$.

Example-3 :

For the following functions, verify Euler's Theorem and find $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x \partial y} + y^2\dfrac{\partial^2 z}{\partial y^2}$.

(1) $z = x^n \log\left(\dfrac{y}{x}\right)$ SPU, Sep. 2014, Nov. 2010

(2) $z = \sin^{-1}\left(\dfrac{x}{y}\right) + \tan^{-1}\left(\dfrac{y}{x}\right)$

l^n. :

) Clearly, z is a homogeneous function of degree n.

$$\frac{\partial z}{\partial x} = nx^{n-1}\log\left(\frac{y}{x}\right) + x^n\,\frac{x}{y}\left(-\frac{y}{x^2}\right) = nx^{n-1}\log\left(\frac{y}{x}\right) - x^{n-1}$$

$$\Rightarrow \quad x\frac{\partial z}{\partial x} = nx^n\log\left(\frac{y}{x}\right) - x^n$$

Also, $\quad \dfrac{\partial z}{\partial y} = x^n\left(\dfrac{x}{y}\right)\left(\dfrac{1}{x}\right) = \dfrac{x^n}{y} \quad \Rightarrow \quad y\dfrac{\partial z}{\partial y} = x^n.$

Hence, $\quad x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = nx^n\log\left(\dfrac{y}{x}\right) = nz.$

Thus Euler's Theorem is verified.

By the Corollary-1,

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)\,z$$

(2) Replacing x by tx and y by ty, $f(tx, ty) = \sin^{-1}\left(\dfrac{x}{y}\right) + \tan^{-1}\left(\dfrac{y}{x}\right) = t^0 f(x, y)$.

Thus $z = f(x, y)$ is a homogeneous function of degree 0.

Now, $$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{1}{y}\right) + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$\Rightarrow \quad x\frac{\partial z}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2}$$

Also, $$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{-x}{y^2}\right) + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$\Rightarrow \quad y\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

hence, $\quad x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$

Thus, Euler's Theorem is verified.

By the Corollary-1,

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)\,z = 0 \qquad (\because\ n = 0)$$

■ **Example-4 :**

If $u = \sin^{-1}\left(\dfrac{x^2y^2}{x+y}\right)$, then prove the following

(1) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3\tan u$

(2) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 3\tan u\,(3\sec^2 u - 1)$

SPU, April-2015, Nov. 2013, 2010

Sol^n. :

Here $u = \sin^{-1}\left(\frac{x^2y^2}{x+y}\right)$ is not a homogeneous function of x, y. Writing the given equation differently, we have $\sin u = \frac{x^2y^2}{x+y}$. Let $z = \varphi(u) = \sin u$. Then $z = \frac{x^2y^2}{x+y}$, which is homogeneous of degree 3. Hence by Corollary-2, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\frac{\phi(u)}{\phi'(u)} = 3\frac{\sin u}{\cos u} = 3\tan u$, which proves (1). Also, by Corollary-3, we have,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \psi(u)[\psi'(u) - 1]$$

where, $\psi(u) = n\frac{\phi(u)}{\phi'(u)} = 3\frac{\sin u}{\cos u} = 3\tan u$

$$\Rightarrow \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 3\tan u\,[3\sec^2 u - 1].$$

THEOREM ON TOTAL DIFFERENTIALS

Throughout this section we consider only those functions of two variables that admit continuous partial derivatives on their domain of definition. That is, if we are discussing about function $z = f(x, y)$, then f_x, f_y exist and are continuous on the domain of f.

Theorem-2 :

Let $z = f(x, y)$ be defined on E. Then $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

DIFFERENTIATION OF COMPOSITE FUNCTIONS

In this section we shall study the differentiation of composite functions. Let $z = f(x, y)$ be function defined on $E \subset R^2$. In turn one can have $x = \phi(t)$ and $y = \psi(t)$, $t \in F \subset R$. This makes f a function of one independent variable t. That is,

$$t \in F \to (\phi(t), \psi(t)) \in E \to f(\phi(t), \psi(t))$$

The following theorem describes the differentiation of f with respect to t in this situation.

Theorem-3 :

Let $z = f(x, y)$ be function defined on $E \subset R^2$ and $x = \phi(t)$, $y = \psi(t)$, $t \in F \subset R$.

Then prove that $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.

Theorem :

Let $z = f(x, y)$ and $u = u(x, y)$, $v = v(x, y)$ then

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}.$$

To extend Theorem-3 for functions of three variables, let $u = f(x, y, z)$ be a function of three variables with $x = x(t)$, $y = y(t)$ and $z = z(t)$. Then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}.$$

CHANGE OF VARIABLES

Like the composite functions we can also consider the following situation. Let $z = f(x, y)$ be function defined on $E \subset R^2$ and let there be another domain $F \subset R^2$ such that for each $(x, y) \in E$, $x = f(u, v)$, $y = y(u, v) \in F \subset R^2$. This is nothing but the change of variable. In this case, the following theorem describes the partial derivatives of f with respect to u and v.

Now we prove Euler's Theorem for three variables. The homogeneous functions of more than two variables are defined as in Definition-5. More explicitly, a function $H = f(x_1, x_2, ..., x_n)$ of n variables is called *homogeneous* if there exists $r \in R$ such that $f(tx_1, tx_2, ..., tx_n) = t^r f(x_1, x_2, ..., x_n)$ for all $t \in R$. In this case, the degree of homogeneity of H is r.

■ **Theorem-4 :**

State and prove Euler's Theorem for function of three variables.

SPU, September-2014

Statement :

Let $H = f(x, y, z)$ be a real value homogeneous function of three variables x, y, z of degree n defined on $E \subset R^3$. If f_x, f_y, f_z exist on E, then prove that

$$x\frac{\partial H}{\partial x} + y\frac{\partial H}{\partial y} + z\frac{\partial H}{\partial z} = nH$$

Proof :

Since $H = f(x, y, z)$ is homogenous function of degree n,

$$H = x^n\varphi\left(\frac{y}{x}, \frac{z}{x}\right) = x^n\varphi(u, v),$$

where $u = \frac{y}{x}$ and $v = \frac{z}{x}$.

$$\text{Hence,}\quad \frac{\partial H}{\partial x} = nx^{n-1}\varphi(u, v) + x^n\left[\frac{\partial\varphi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\varphi}{\partial v}\frac{\partial v}{\partial x}\right]$$

$$= nx^{n-1}\varphi(u, v) + x^n\left[-\frac{y}{x^2}\frac{\partial\varphi}{\partial u} - \frac{z}{x^2}\frac{\partial\varphi}{\partial v}\right]$$

$$= nx^{n-1}\varphi(u, v) - x^{n-2}\,y\frac{\partial\varphi}{\partial u} - x^{n-2}\,z\frac{\partial\varphi}{\partial v}$$

$$\Rightarrow \quad x\frac{\partial H}{\partial x} = nx^n\,\varphi(u,v) - x^{n-1}\,y\frac{\partial \varphi}{\partial u} - x^{n-1}\,z\frac{\partial \varphi}{\partial v} \qquad \ldots\ldots\ldots\ldots (9)$$

Now, $\frac{\partial H}{\partial y} = x^n\left[\frac{\partial \varphi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \varphi}{\partial v}\frac{\partial v}{\partial y}\right] = x^n\left[\frac{1}{x}\frac{\partial \varphi}{\partial u} + 0\,\frac{\partial \varphi}{\partial v}\right] = x^{n-1}\frac{\partial \varphi}{\partial u}$

$$\Rightarrow \quad y\frac{\partial H}{\partial y} = x^{n-1}\,y\frac{\partial \varphi}{\partial u}. \qquad \ldots\ldots\ldots\ldots (10)$$

Similarly,

$$z\frac{\partial H}{\partial z} = x^{n-1}\,z\frac{\partial \varphi}{\partial v} \qquad \ldots\ldots\ldots\ldots (11)$$

Adding (9), (10) and (11) we have,

$$x\frac{\partial H}{\partial x} + y\frac{\partial H}{\partial y} + z\frac{\partial H}{\partial z} = nx^n\varphi(u,v) = nH$$

This completes the proof.

A noted in case of the function of two variables, here also we recall that the converse of Euler's Theorem also holds. That is, if a function $z = f(x, y)$ satisfies (8), on a certain domain, then it must be homogeneous on that domain.

■ **Example-5 :**

Find $\frac{dz}{dt}$ when $z = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$. Also verify by the direct substitution.

SPU, December-2015, September-2014

Sol^n. :

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \frac{1}{\sqrt{1-(x-y)^2}}\cdot 3 - \frac{1}{\sqrt{1-(x-y)^2}}\cdot 12t^2$$

$$= \frac{3(1-4t^2)}{\sqrt{1-(x-y)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}}$$

On the other hand, verifying directly by putting the values of x and y in z, we have $z = \sin^{-1}(3t - 4t^3)$.

$$\Rightarrow \quad \frac{dz}{dt} = \frac{(3-12t^2)}{\sqrt{1-(3t-4t^3)^2}} = \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}}$$

■ **Example-6 :**

If $z = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, then prove that

$$\left[\frac{\partial z}{\partial x}\right]^2 + \left[\frac{\partial z}{\partial y}\right]^2 = \left[\frac{\partial z}{\partial r}\right]^2 + \frac{1}{r^2}\left[\frac{\partial z}{\partial \theta}\right]^2$$

SPU, April-2016, 2015, Nov. 2010

Sol^n. :

Here x, y are functions of r, θ. Hence z is a composite function of r, θ.

Thus, $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}$

$$\Rightarrow \quad \left[\frac{\partial z}{\partial r}\right]^2 = \cos^2\theta\left[\frac{\partial z}{\partial x}\right]^2 + 2\sin\theta\cos\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + \sin^2\theta\left[\frac{\partial z}{\partial y}\right]^2 \qquad \ldots\ldots\ldots\ldots (12)$$

Also

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial z}{\partial y}$$

$$\Rightarrow \quad \left[\frac{\partial z}{\partial \theta}\right]^2 = r^2\sin^2\theta\left[\frac{\partial z}{\partial x}\right]^2 - 2r^2\sin\theta\cos\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + r^2\cos^2\theta\left[\frac{\partial z}{\partial y}\right]^2$$

$$\Rightarrow \quad \frac{1}{r^2}\left[\frac{\partial z}{\partial \theta}\right]^2 = \sin^2\theta\left[\frac{\partial z}{\partial x}\right]^2 - 2\sin\theta\cos\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + \cos^2\theta\left[\frac{\partial z}{\partial y}\right]^2. \qquad \ldots\ldots\ldots\ldots (13)$$

Adding (12) and (13) we get,

$$\left[\frac{\partial z}{\partial r}\right]^2 + \frac{1}{r^2}\left[\frac{\partial z}{\partial \theta}\right]^2 = \left[\frac{\partial z}{\partial x}\right]^2 + \left[\frac{\partial z}{\partial y}\right]^2$$

■ **Example-7 :**

If $H = f(2x - 3y, 3y - 4z, 4z - 2x)$, then prove that $\frac{1}{2}\frac{\partial H}{\partial x} + \frac{1}{3}\frac{\partial H}{\partial y} + \frac{1}{4}\frac{\partial H}{\partial z} = 0$.

SPU, November-2013

Sol^n. :

Let $u = 2x - 3y$, $v = 3y - 4z$, $w = 4z - 2x$. Then $H = f(u, v, w)$. Hence H is a composite function of x, y, z.

Therefore,

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial H}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial H}{\partial w}\frac{\partial w}{\partial x} = 2\frac{\partial H}{\partial u} + 0\frac{\partial H}{\partial v} - 2\frac{\partial H}{\partial w}$$

$$= 2\frac{\partial H}{\partial u} - 2\frac{\partial H}{\partial w} \quad \text{... ... ... (14)}$$

Also,

$$\frac{\partial H}{\partial y} = \frac{\partial H}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial H}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial H}{\partial w}\frac{\partial w}{\partial y} = -3\frac{\partial H}{\partial u} + 3\frac{\partial H}{\partial v} + 0\frac{\partial H}{\partial w}$$

$$= -3\frac{\partial H}{\partial u} + 3\frac{\partial H}{\partial v} \quad \text{... ... ... (15)}$$

Finally,

$$\frac{\partial H}{\partial z} = \frac{\partial H}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial H}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial H}{\partial w}\frac{\partial w}{\partial z} = 0\frac{\partial H}{\partial u} - 4\frac{\partial H}{\partial v} + 4\frac{\partial H}{\partial w}$$

$$= -4\frac{\partial H}{\partial v} + 4\frac{\partial H}{\partial w} \quad \text{... ... ... (16)}$$

Hence,

$$\frac{1}{2}\frac{\partial H}{\partial x} + \frac{1}{3}\frac{\partial H}{\partial y} + \frac{1}{4}\frac{\partial H}{\partial z} = \frac{\partial H}{\partial u} - \frac{\partial H}{\partial w} - \frac{\partial H}{\partial u} + \frac{\partial H}{\partial v} - \frac{\partial H}{\partial v} + \frac{\partial H}{\partial w} = 0.$$

■ **Example-8 :**

If $z = f(x, y)$ and $u = e^x \cos y$, $v = e^x \sin y$. Then prove that $\frac{\partial f}{\partial x} = u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v}$.

Sol^n. :

$u = e^x \cos y$, $v = e^x \sin y$.

Hence,

$$u^2 + v^2 = e^{2x} \Rightarrow e^x = \sqrt{u^2 + v^2}$$

$$\Rightarrow \quad x = \frac{1}{2}\log(u^2 + v^2)$$

Also,

$$\frac{v}{u} = \tan y \quad \Rightarrow \quad y = \tan^{-1}\left(\frac{v}{u}\right).$$

Thus, x, y are functions of u, v and so, z is a composite function of u, v.

Also, $\frac{\partial x}{\partial u} = \frac{1}{2}\cdot\frac{2u}{(u^2+v^2)} = \frac{u}{u^2+v^2}$

and $\frac{\partial y}{\partial u} = \frac{1}{1+\frac{v^2}{u^2}}\left(\frac{-v}{u^2}\right) = \frac{u^2}{u^2+v^2}\cdot\left(\frac{-v}{u^2}\right) = \frac{-v}{u^2+v^2}$

Now,

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial f}{\partial x}\left[\frac{u}{u^2+v^2}\right] + \frac{\partial f}{\partial y}\left[\frac{-v}{u^2+v^2}\right]$$

$$\Rightarrow \quad u\frac{\partial f}{\partial u} = \left[\frac{u^2}{u^2+v^2}\right]\frac{\partial f}{\partial x} - \left[\frac{uv}{u^2+v^2}\right]\frac{\partial f}{\partial y} \quad \text{... ... ... (17)}$$

Similarly,

$$v\frac{\partial f}{\partial v} = \left[\frac{v^2}{u^2+v^2}\right]\frac{\partial f}{\partial x} + \left[\frac{uv}{u^2+v^2}\right]\frac{\partial f}{\partial y} \quad \text{... ... ... (18)}$$

Adding (17) and (18) we get, $u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}$

DIFFERENTIATION OF IMPLICIT FUNCTIONS

Many a times we are given an expression $f(x, y) = c$, where $c \in R$ is a constant. Note here that, x and y are associated by a rule however we may not be able to write y as a function of x. In this case, we say that y is a function of x, implicitly described by $f(x, y) = c$ or y is an **implicit function** of x. We obtain the method of calculating $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ using the tools of partial derivatives.

■ **Theorem-5 :**

Let a function y of x be implicitly described by $f(x, y) = c$. Then prove that

(1) $\frac{dy}{dx} = -\frac{f_x}{f_y}$ **SPU, April-2015, Nov. 2011, 2010**

(2) $\frac{d^2y}{dx^2} = -\frac{f_{xx}(f_y)^2 - 2f_{xy}f_xf_y + f_{yy}(f_x)^2}{(f_y)^3}$ **SPU, November-2010**

Proof :

(1) We know that f is a function of x and y. Also, y is an implicit function of x. So, f is a composite function of x. Hence differentiating the equation $f(x, y) = c$ with respect to x, we get

$$\frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{\partial x} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}$$

This proves (1).

(2) $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{-f_x}{f_y}\right)$

$$= -\frac{f_y \frac{d}{dx}(f_x) - f_x \frac{d}{dx}(f_y)}{(f_y)^2}$$

$$= -\frac{f_y\left(\frac{\partial}{\partial x}(f_x) + \frac{\partial}{\partial y}(f_x)\frac{dy}{dx}\right) - f_x\left(\frac{\partial}{\partial x}(f_y) + \frac{\partial}{\partial y}(f_y)\frac{dy}{dx}\right)}{(f_y)^2}$$

$$= -\frac{f_y\left(f_{xx} + f_{xy}\left(-\frac{f_x}{f_y}\right)\right) - f_x\left(f_{yx} + f_{yy}\left(-\frac{f_x}{f_y}\right)\right)}{(f_y)^2}$$

$$= -\frac{f_{xx}(f_y)^2 - f_y f_x f_{xy} - f_x f_y f_{yx} + f_{yy}(f_x)^2}{(f_y)^3}$$

$$= -\frac{f_{xx}(f_y)^2 - 2f_y f_x f_{xy} + f_{yy}(f_x)^2}{(f_y)^3}$$

■ **Example-9 :**

Find $\frac{dy}{dx}$ when

(1) $x\sin(x - y) - (x + y) = 0$ **SPU, September-2014**

(2) $x^y = y^x$ **SPU, November-2010**

Proof :

(1) Let $f(x, y) = x\sin(x - y) - (x + y)$.

Since $f(x, y) = 0$, by the previous theorem, we have,

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{x\cos(x-y) + \sin(x-y) - 1}{x\cos(x-y)(-1) - 1}$$

$$= \frac{x\cos(x-y) + \sin(x-y) - 1}{x\cos(x-y) + 1}$$

(2) Let $f(x, y) = x^y - y^x$.

Since $f(x, y) = 0$, by the previous theorem, we have,

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{yx^{y-1} - y^x\log y}{x^y\log x - xy^{x-1}} = \frac{y^x\log y - yx^{y-1}}{x^y\log x - xy^{x-1}}$$

■ **Example-10 :**

If $z = xy\,f\left(\frac{y}{x}\right)$ and z is constant, then show that $\frac{f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)} = \frac{x\left[y + x\frac{dy}{dx}\right]}{y\left[y - x\frac{dy}{dx}\right]}$

SPU, April-2016, December-2015, April-2015

Sol^n. :

Let $F(x, y) = xy\,f\left(\frac{y}{x}\right)$. Then $F(x, y) = z$, z is constant. Thus y is an implicit function of x.

So, $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$ ... ... (19)

Now, differentiating $F(x, y)$ with respect to x, we get,

$$\frac{\partial F}{\partial x} = y\,f\left(\frac{y}{x}\right) + xy\,f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$= y\,f\left(\frac{y}{x}\right) - \frac{y^2}{x}f'\left(\frac{y}{x}\right)$$

$$= \frac{y}{x}\left[x\,f\left(\frac{y}{x}\right) - y\,f'\left(\frac{y}{x}\right)\right]$$

Similarly,

$$\frac{\partial F}{\partial y} = x\,f\left(\frac{y}{x}\right) + xy\,f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = x\,f\left(\frac{y}{x}\right) + y\,f'\left(\frac{y}{x}\right).$$

Putting these values in (19), we have

$$\frac{y}{x}\left[x\,f\left(\frac{y}{x}\right) - y\,f'\left(\frac{y}{x}\right)\right] + \left[x\,f\left(\frac{y}{x}\right) + y\,f'\left(\frac{y}{x}\right)\right]\frac{dy}{dx} = 0$$

$$\Rightarrow \left[y + x\frac{dy}{dx}\right]f\left(\frac{y}{x}\right) = \frac{y}{x}\left[y - x\frac{dy}{dx}\right]f'\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)} = \frac{x}{y}\left[\frac{y + x\frac{dy}{dx}}{y - x\frac{dy}{dx}}\right].$$

■ **Example-11 :**

If A, B and C are angles of a Δ ABC such that $\sin^2 A + \sin^2 B + \sin^2 C = K$, a constant, then prove that $\frac{dB}{dC} = \frac{\tan C - \tan A}{\tan A - \tan B}$.

SPU, April-2016, December-2015, Sep. 2014, Nov. 2013, 2011

Sol^n. :

Clearly, A + B + C = π. So, A = π − (B + C). Therefore, sinA = sin(B + C). Let $f(B, C) = \sin^2(B+C) + \sin^2 B + \sin^2 C - K$. Hence $f(B, C) = 0$, i.e., B is an implict function of C. So, $\frac{dB}{dC} = -\frac{f_C}{f_B}$. Also,

$$\begin{aligned} f_B &= \frac{\partial f}{\partial B} \\ &= 2\sin(B+C)\cos(B+C) + 2\sin B\cos B \\ &= \sin 2(B+C) + \sin 2B \\ &= \sin(2\pi - 2A) + \sin 2B \\ &= -\sin 2A + \sin 2B \\ &= 2\cos(B+A)\sin(B-A) \\ &= 2\cos(\pi - C)\sin(B-A) \\ &= -2\cos C\sin(B-A) \\ &= 2\cos C\sin(A-B) \end{aligned}$$

Similarly, we get,

$$f_C = 2\cos B\sin(A-C)$$

Hence,

$$\begin{aligned} \frac{dB}{dC} &= -\frac{\cos B\sin(A-C)}{\cos C\sin(A-B)} \\ &= -\frac{\cos B(\sin A\cos C - \cos A\sin C)}{\cos C(\sin A\cos B - \cos A\sin B)} \\ &= -\frac{\sin A\cos B\cos C - \cos A\cos B\sin C}{\sin A\cos B\cos C - \cos A\sin B\cos C} \end{aligned}$$

Dividing by cosA cosB cosC, we get,

$$\frac{dB}{dC} = -\frac{\tan A - \tan C}{\tan A - \tan B} = \frac{\tan C - \tan A}{\tan A - \tan B}.$$

■ **Example-12 :**

Check whether $z = \frac{4x^2y^2 + 5xy^3}{\sqrt{x} + \sqrt{y}}$ is homogeneous or not ? If so find its degree.

SPU, April-2015

Sol^n. :

$$\text{Here } z = \frac{x^4\left[4\left(\frac{y}{x}\right)^2 + 5\left(\frac{y}{x}\right)^3\right]}{\sqrt{x}\left[1 + \sqrt{\frac{y}{x}}\right]}$$

$$= x^{4-\frac{1}{2}} f\left(\frac{y}{x}\right)$$

$$= x^{\frac{7}{2}} f\left(\frac{y}{x}\right)$$

$\therefore$ Given function is a homogeneous function of degree $\frac{7}{2}$.

■ **Example-13 :**

Verify Euler's theorem for the following

(1) $z = 3x^2y - 4xy^2$.

SPU, April-2015

Sol^n. :

$$z = x^3\left[3\left(\frac{y}{x}\right) - 4\left(\frac{y}{x}\right)^2\right] = x^3 f\left(\frac{y}{x}\right)$$

which is homogeneous function of degree 3.

$\therefore$ By Euler's theorem $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$

$$\begin{aligned} \text{L.H.S.} &= x(6xy - 4y^2) + y(3x^2 - 8xy) \\ &= 6x^2y - 4xy^2 + 3x^2y - 8xy^2 \\ &= 9x^2y - 12xy^2 \\ &= 3[3x^2y - 4xy^2] \\ &= 3z = \text{R.H.S.} \end{aligned}$$

Euler's theorem is verified.

2) $u = \sin^{-1}\left(\frac{x}{y}\right)$ SPU, September-2014

Sol^n. :

$$u = \sin^{-1}\left(\frac{x}{y}\right) \Rightarrow \sin u = \frac{x}{y}$$

Let, $z = \sin u$, than $z = \frac{x}{y}$

Which is homogeneous function of degree 0. Let $\phi(u) = z$

By corollary of Euler's theorem we say that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{\phi(u)}{\phi'(u)}$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \qquad (\because n = 0)$$

$$\text{L.H.S.} = x \cdot \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{1}{y}\right) + y\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\cdot\left(-\frac{x}{y^2}\right)$$

$$= \frac{x}{\sqrt{y^2-x^2}} - \frac{x}{\sqrt{y^2-x^2}} = 0$$

= R.H.S. Hence Euler's theorem is verified.

3) $z = x^2y - xy^2$. SPU, November-2013

Sol^n. :

Here $z = x^3\left[\frac{y}{x} - \left(\frac{y}{x}\right)^2\right]$ is a homogeneous function of degree 3.

$\therefore$ By Euler's theorem, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$

$$\therefore \text{L.H.S.} = x[2xy - y^2] + y[x^2 - 2xy]$$
$$= 2x^2y - xy^2 + x^2y - 2xy^2$$
$$= 3[x^2y - xy^2]$$
$$= 3z = \text{R.H.S.}$$

Thus Euler's theorem is verified.

■ **Example-14 :**

Determine whether the function $f(x, y) = \frac{\sqrt{x} - \sqrt{y}}{x + y}$ is homogeneous or not ?

SPU, September-2014

Sol^n. :

$$\text{Here } f(x, y) = \frac{\sqrt{x}\left[1 - \sqrt{\frac{y}{x}}\right]}{x\left[1 + \frac{y}{x}\right]} = x^{-1/2} f\left(\frac{y}{x}\right)$$

$\therefore$ It is a homogeneous function of degree $-\frac{1}{2}$.

■ **Example-15 :**

If $u = \frac{x^3 + y^3}{xy}$ then find $x\frac{\partial y}{\partial x} + y\frac{\partial y}{\partial y}$. SPU, November-2013

Sol^n. :

$$\text{Here, } u = \frac{x^3 + y^3}{xy} = \frac{x^3\left[1 + \left(\frac{y}{x}\right)^3\right]}{x^2\left(\frac{y}{x}\right)} = x f\left(\frac{y}{x}\right)$$

$\therefore$ If is a homogeneous function of degree 1.

$\therefore$ By Euler's theorem $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 1u = u.$

■ **Example-16 :**

For $u = x^3 - 3xy^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. SPU, November-2013

Sol^n. :

$$\frac{\partial y}{\partial x} = 3x^2 - 3y^2, \; \frac{\partial u}{\partial y} = -6xy$$

$$\frac{\partial^2 u}{\partial x^2} = 6x; \; \frac{\partial^2 u}{\partial y^2} = -6x$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0.$$

MULTIPLE CHOICE QUESTIONS

■ **Fill in the Blanks :**

1. If $u = x^3 - 3xy^2$, then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ = ______.

(a) 0 (b) 1 (c) 2 (d) -1

2. If $u = x^3 - 3xy^2$, then $\frac{\partial^2 u}{\partial x \partial y}$ = ______.

(a) $6y$ (b) $-6y$ (c) $6x$ (d) $-6x$

3. If $f(x, y) = \frac{x-y}{x+y}$, then f_x = ______.

(a) $\frac{2x}{(x+y)^2}$ (b) $\frac{-2y}{(x+y)^2}$ (c) $\frac{2y}{(x+y)^2}$ (d) $\frac{-2x}{(x+y)^2}$

4. If $f(x, y) = \frac{x-y}{x+y}$, then f_y = ______.

(a) $\frac{2x}{(x+y)^2}$ (b) $\frac{-2y}{(x+y)^2}$ (c) $\frac{2y}{(x+y)^2}$ (d) $\frac{-2x}{(x+y)^2}$

5. $f(x, y) = \frac{\sqrt[5]{x} - \sqrt[5]{y}}{x^3 + y^3}$ is a homogeneous function of degree ______.

(a) $-\frac{14}{5}$ (b) $\frac{5}{3}$ (c) $\frac{14}{5}$ (d) $\frac{5}{6}$

6. Let $u = u(x, y)$ be a homogeneous function and $z = \phi(u)$ be a homogeneous function of degree n. Then $x\frac{\partial y}{\partial x} + y\frac{\partial u}{\partial y}$ = ______.

(a) $\frac{\phi(u)}{\phi'(u)}$ (b) $n\frac{\phi(u)}{\phi'(u)}$ (c) $n\phi(u)$ (d) nz

7. If $z = x^n \log\left(\frac{y}{x}\right)$ then $y\frac{\partial z}{\partial y}$ = ______.

(a) $\frac{x^n}{y}$ (b) $\frac{x^{n-1}}{y}$ (c) x^n (d) x^{n-1}

8. $z = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$ is a homogeneous function of degree ______.

(a) 1 (b) 2 (c) -1 (d) 0

9. If $z = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$ then $x\frac{\partial z}{\partial x}$ = ______.

(a) $-y\frac{\partial z}{\partial y}$ (b) $y\frac{\partial z}{\partial y}$ (c) $\frac{\partial z}{\partial y}$ (d) 0

10. If $z = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$ then $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2}$ = ______.

(a) 1 (b) 0 (c) $n(n-1)z$ (d) nz

11. If $u = \sin^{-1}\left[\frac{x^2 y^2}{x+y}\right]$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ = ______.

(a) $\tan u$ (b) $2\tan u$ (c) $3\tan u$ (d) $2\cot u$

12. If $z = \sin^{-1}[3t - 4t^3]$ then $\frac{dz}{dt}$ = ______.

(a) $\frac{3}{1-t^2}$ (b) $1-t^2$ (c) $\sqrt{\frac{3}{1-t^2}}$ (d) $\frac{3}{\sqrt{1-t^2}}$

13. If $z = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$ then $\left(\frac{\partial z}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2$ + ______.

(a) $\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$ (b) $\left(\frac{\partial z}{\partial \theta}\right)^2$ (c) $\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}$ (d) $\frac{1}{r}\left(\frac{\partial z}{\partial \theta}\right)^2$

14. If $u = e^x \cos y$; $v = e^x \sin y$ then x = ______.

(a) $\log(u^2 + v^2)$ (b) $\frac{1}{2}\log(u^2 + v^2)$ (c) $\tan^{-1}\left(\frac{v}{u}\right)$ (d) $\tan^{-1}\left(\frac{u}{v}\right)$

15. If $u = e^x \cos y$; $v = e^x \sin y$ then y = ______.

(a) $\log(u^2 + v^2)$ (b) $\frac{1}{2}\log(u^2 + v^2)$ (c) $\tan^{-1}\left(\frac{v}{u}\right)$ (d) $\tan^{-1}\left(\frac{u}{v}\right)$

16. If $f(x, y) = C$ is implicit function then $\frac{dy}{dx}$ = ______.

(a) $\frac{f_x}{f_y}$ (b) $-\frac{f_y}{f_x}$ (c) $\frac{f_y}{f_x}$ (d) $-\frac{f_x}{f_y}$

17. $f(x, y) = x^2y^4 - x^3y^3 + xy^5$ is a homogeneous function of degree ______.

(a) 6 (b) 4 (c) 5 (d) 0

18. $f(x, y) = x^4y^4 - x^5y^3 + x^6y^2$ is a homogeneous function of degree ______.

(a) 4 (b) 8 (c) 0 (d) 6

19. If $f(x, y) = C$ is a implicit function then $f_y \frac{dy}{dx}$ = ______.

(a) f_x (b) 0 (c) $-f_x$ (d) $-f_y$

20. The Euler's theorem is defined for the functions which are ______.

(a) Continuous (b) Differentiable (c) Homogeneous (d) None of the above

21. If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is ______.

(a) $-\cot u$ (b) $-\frac{1}{2}\cot u$ (c) $-\frac{1}{2}\tan u$ (d) None of the above

22. If $u = \log\left(\frac{x^2+y^2}{x+y}\right)$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is ______.

(a) -1 (b) e^u (c) 1 (d) None of the above

23. If $u = \log\left(\frac{x^2+y^2}{x+y}\right)$ then $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$ is ______.

(a) -1 (b) e^u (c) 1 (d) 0

24. Degree of homogeneity of $u = 3x^2yz + 5xy^2z + 4z^4$ is ______.

(a) 2 (b) 3 (c) 4 (d) 0

25. If $u = f(x-y, y-z, z-x)$, then $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$ = ______.

(a) 1 (b) 2 (c) 3 (d) 0

26. A function $u = \sin\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$ is given then ______.

(a) u is homogeneous function

(b) u is non-homogeneous but $\sin^{-1} u$ is homogeneous function

(c) Both u and $\sin^{-1}$ are homogeneous function

(d) None of the above

ANSWERS

1. (a),	2. (b),	3. (c),	4. (d),	5. (a),	6. (b),	7. (c),	8. (d),
9. (a),	10. (b),	11. (c),	12. (d),	13. (a),	14. (b),	15. (c),	16. (d),
17. (a),	18. (b),	19. (c),	20. (c),	21. (b),	22. (c),	23. (d),	24. (c),
25. (d),	26. (c).						

SHORT QUESTIONS

1. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ for the following. Also check $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

(a) $u = \frac{1}{\sqrt{x^2+y^2}}$

(b) $u = \frac{x}{\sqrt{x^2+y^2}}$

(c) $u = \log(x^2+y^2)$

(d) $u = \log(x + \sqrt{x^2+y^2})$

(e) $\tan u = \frac{x^2+y^2}{x+y}$

(f) $u = x^y + y^x$

(g) $u = (x^2+2)(y^2-3y+6)$

(h) $u = \frac{xy}{\sqrt{1+x^2+y^2}}$

(i) $u = \log(x\sin y + y\sin x)$

(j) $u = \log\left(\tan\frac{y}{x}\right)$

(k) $u = \tan^{-1}\left(\frac{xy}{\sqrt{1+x^2+y^2}}\right)$

(l) $u = \log\left(\frac{x^2+y^2}{xy}\right)$

(m) $u = y^{-1/2}e^{-(x-a)^2/4y}$

2. **Prove that** $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$, **when**

(a) $u=\tan^{-1}\left(\dfrac{y}{x}\right)$

(b) $u=\log(x^2+y^2)$

(c) $u=e^{ay}\cos ax$

(d) $u=e^x(x\cos y-y\sin y)$

(e) $u=2(ax+by)^2-x^2-y^2$, where $a^2+b^2=1$

(f) $u=a\cos(bx+c)\cosh(by+d)$, where a, b, c, d are constant.

3. **Prove the following for** $x=r\cos\theta,\ y=r\sin\theta$.

(a) $\dfrac{\partial x}{\partial r}=\dfrac{\partial r}{\partial x}$

(b) $\dfrac{\partial x}{\partial \theta}=r^2\dfrac{\partial \theta}{\partial x}$

4. **Find** $\dfrac{dy}{dx}$ **for the following.**

(a) $e^x+e^y=2xy$

(b) $ax^2+2hxy+by^2=1$

(c) $x^y+y^x=(x+y)^{x+y}$

(d) $x^y+y^x=4a^2xy$

(e) $x^y+y^x+x^x+y^y=0$

(f) $x^y+y^x=(\sin x)^y$

(g) $(\cos x)^y-(\sin y)^x=0$

5. **Find** $\dfrac{dz}{dt}$ **for the following :**

(a) $z=x^2+y^2,\ x=at^2,\ y=2at$

(b) $z=\tan^{-1}\left(\dfrac{x}{y}\right),\ x=2t,\ y=1-t^2$

(c) $z=xe^y,\ x=2t,\ y=1-t^2$

(d) $z=\cosh\left(\dfrac{y}{x}\right),\ x=t^2,\ y=e^t$.

EXERCISE

1. If $u=x^y$, then prove that $\dfrac{\partial^3 u}{\partial x^2\partial y}=\dfrac{\partial^3 u}{\partial x\partial y\partial x}$.

2. If $u=\dfrac{xy}{2x+z}$, then prove that $\dfrac{\partial^3 u}{\partial y\partial z^2}=\dfrac{\partial^3 u}{\partial z^2\partial y}$.

3. If $u=x^m y^n$, then prove that $\dfrac{\partial^3 u}{\partial x\partial y\partial x}=\dfrac{\partial^3 u}{\partial y\partial x^2}$.

4. For $u=e^{xyz}$, find all third order partial derivatives and compare them.

5. If $u=\sqrt{x^2+y^2}$, then find $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}$.

6. If $u^2=x^2+y^2+z^2$, then prove that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=\dfrac{2}{u}$.

7. If $u=(x^2+y^2+z^2)^{-1/2}$, then prove

(a) $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=-u$

(b) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=0$

8. If $u=\log r$, where $r^2=(x-a)^2+(y-b)^2+(z-c)^2$, then prove that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=\dfrac{1}{r^2}$.

9. If $u=3(ax+by+cz)^2-(x^2+y^2+z^2)$ and if $a^2+b^2+c^2=1$, then prove that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=0$.

10. Prove that following for $x=r\cos\theta,\ y=r\sin\theta$ then prove that $\dfrac{\partial^2 r}{\partial x^2}+\dfrac{\partial^2 r}{\partial y^2}=\dfrac{1}{r^2}\left[\left(\dfrac{\partial r}{\partial x}\right)^2+\left(\dfrac{\partial r}{\partial y}\right)^2\right]$.

11. If $x=\cos\theta-r\sin\theta,\ y=\sin\theta+r\cos\theta$, then prove the following.

(a) $\dfrac{\partial\theta}{\partial x}=-\dfrac{\cos\theta}{r}$

(b) $\dfrac{\partial^2\theta}{\partial x^2}=\dfrac{\cos\theta}{r^2}(\cos\theta-2r\sin\theta)$

(c) $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}$

12. If $u=\log(x^3+y^3+z^3-3xyz)$, then prove the following.

(a) $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=\dfrac{3}{x+y+z}$

(b) $\left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}+\dfrac{\partial}{\partial z}\right)^2 u=-\dfrac{9}{(x+y+z)^2}$

13. If $u=\dfrac{1}{\sqrt{y^2-2xy+1}}$, then prove that $\dfrac{\partial}{\partial x}\left((1-x^2)\dfrac{\partial u}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(y^2\dfrac{\partial u}{\partial y}\right)=0$.

14. If $x^x y^y z^z=a$, then prove that $\dfrac{\partial^2 z}{\partial x\partial y}=-(x\log(ex))^{-1}$, when $x=y=z$.

15. If $x^x y^y z^z=c$, then find $\dfrac{\partial^2 z}{\partial x^2}-2xy\dfrac{\partial^2 z}{\partial x\partial y}+\dfrac{\partial^2 z}{\partial y^2}$, when $x=y=z=1$.

16. If $z=\dfrac{x^2+y^2}{x+y}$, then prove that $\left(\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}\right)^2=4\left(1-\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}\right)$.

17. If $u=\log\left(\sqrt{x^2+y^2+z^2}\right)$, then find $(x^2+y^2+z^2)\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}\right)$.

18. If $u=f(r)$, where $r^2=x^2+y^2$, then prove that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=f''(r)+\dfrac{1}{r}f'(r)$, where $'$ denotes the derivative with respect to r.

19. Let $u=f(r)$, $r^2=x^2+y^2+z^2$. Prove that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=f''(r)+\dfrac{2}{r}f'(r)$, where $'$ denotes the derivative with respect to r.

20. Let $\theta=t^n e^{-\frac{r^2}{4t}}$. Find n for which $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\theta}{\partial r}\right)=\dfrac{\partial\theta}{\partial t}$.

21. If $u=f(ax+y)+g(-ax+y)$, then prove that $\dfrac{\partial^2 u}{\partial x^2}-a^2\dfrac{\partial^2 u}{\partial y^2}=0$.

22. Verify Euler's Theorem for the following functions.

(a) $u=xy\,f\left(\dfrac{y}{x}\right)$

(b) $u=\dfrac{xy}{x+y}$

(c) $u=\dfrac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$

(d) $u=\dfrac{x^4-xy^3}{x^3+y^3}$

(e) $u=x^n\sin\dfrac{y}{x}$

(f) $u=\dfrac{1}{\sqrt{x^2+y^2}}$

(g) $u=ax^2+2hxy+by^2$

23. Find $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}$ and $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}$ for the following.

(a) $u=\cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

(b) $u=\log\left(\dfrac{x^2+y^2}{x+y}\right)$

(c) $u=\dfrac{x^2y+xy^2}{x^2+y^2}$

(d) $u=\dfrac{xy}{x+y}$

(e) $u=x\,f\left(\dfrac{y}{x}\right)+g\left(\dfrac{y}{x}\right)$

(f) $u=\left(\dfrac{x}{y}\right)^{y/x}$

(g) $u=\log\left(\dfrac{x^3+y^3}{x^2+y^2}\right)$

(h) $u=\sin^{-1}\left(\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}-y^{1/2}}\right)$

(i) $u=\tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{x+y}\right)$

(j) $u=\operatorname{cosec}^{-1}\left(\sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}\right)$

(k) $u=\sec^{-1}\left(\dfrac{x^2+y^2}{x+y}\right)$

(l) $u=\tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$

(m) $u=\sin^{-1}\left(\dfrac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{-1/5}}\right)$

(n) $e^u=\dfrac{\sqrt{x^2+y^2}}{x+y}$

(o) $u=\log(x^3+y^3-x^2y-xy^2)$

(p) $u=\log r$, where $r^2=x^2+y^2$

(q) $u=x^2\tan^{-1}\left(\dfrac{y}{x}\right)+y^2\sin^{-1}\left(\dfrac{x}{y}\right)$

24. Find $\dfrac{d^2y}{dx^2}$ for the following.

(a) $x^3-3axy+y^3=0$

(b) $x^3-3ax^2+y^3=0$

(c) $x^m+y^m=a^m$

(d) $x^3+y^3=3ax^2y$

25. If $z=f(x,y)$, $x=r\cos\theta$, $y=r\sin\theta$, then prove the following.

(a) $\dfrac{\partial z}{\partial x}=\cos\theta\dfrac{\partial z}{\partial r}-\dfrac{1}{r}\sin\theta\dfrac{\partial z}{\partial\theta}$

(b) $\dfrac{\partial^2}{\partial x\partial y}(r^n\cos n\theta)=-n(n-1)\,r^{n-2}\sin(n-2)\theta$

(c) $\left(\dfrac{\partial z}{\partial x}\right)^2+\left(\dfrac{\partial z}{\partial y}\right)^2=\left(\dfrac{\partial z}{\partial r}\right)^2+\dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial\theta}\right)^2$

(d) $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=\dfrac{\partial^2 z}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial z}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 z}{\partial\theta^2}$

(e) $\dfrac{\partial^2\theta}{\partial x\partial y}=-\dfrac{\cos 2\theta}{r^2}=\dfrac{\partial^2\theta}{\partial y\partial x}$

(f) $\dfrac{\partial^2 r}{\partial x\partial y}=-\dfrac{\sin\theta\cos\theta}{r}=\dfrac{\partial^2 r}{\partial y\partial x}$

26. If $z=f(x,y)$, $x=e^u+e^{-v}$, $y=e^{-u}-e^v$, then prove that $\dfrac{\partial z}{\partial u}-\dfrac{\partial z}{\partial v}=x\dfrac{\partial z}{\partial x}-y\dfrac{\partial z}{\partial y}$.

27. If $z = f(x, y)$, $x = e^u \sec v$, $y = e^u \tan v$, then prove that

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - \cos^2 v\left(\frac{\partial z}{\partial v}\right)^2\right].$$

28. $z = f(x, y)$, $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$, then prove

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \text{ and } \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

29. If $z = f(u, v)$, $u = x^2 - 2xy - y^2$, $v = y$; then show that

$$(x + y)\frac{\partial z}{\partial x} + (x - y)\frac{\partial z}{\partial y} = 0 \text{ is equivalent to } \frac{\partial z}{\partial v} = 0.$$

30. If $u = \sqrt{x^2 + y^2}$ and $x^3 + y^3 + 3axy = 5a^2$, then $\frac{du}{dx}$ at (a, a).

31. Find $\frac{du}{dx}$ for the following.

(a) $u = x\log(xy)$ and $x^3 + y^3 + 3axy = 0$ (c) $u = \sin(x^2 + y^2)$ and $a^2x^2 + b^2y^2 = c^2$

(b) $u = x^2 y$ and $x^2 + xy + y^2 = 1$ (d) $u = e^{ax}(y - z)$, $y = a\sin x$ and $z = \cos x$.

32. Find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$ for the following.

(a) $u = 3x^2yz + 5xy^2z + 4z^4$ (d) $u = \sin^{-1}\left(\frac{x}{y}\right) + \cos^{-1}\left(\frac{y}{z}\right) = \tan^{-1}\left(\frac{z}{x}\right)$

(b) $u = \sin\left(\frac{xy + yz + zx}{x^2 + y^2 + z^2}\right)$ (e) $u = \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)^n$

(c) $u = \frac{x - y - z}{x^2 + y^2 + z^2}$

33. $u = f(x - y, y - z, z - x)$, then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

34. If $u = f(x + ay) + g(x - ay)$, then prove that $\frac{\partial^2 u}{\partial y^2} = a^2\frac{\partial^2 u}{\partial x^2}$

35. Let the change of axis be given by $x = u\cos\alpha - v\sin\alpha$ and $y = u\sin\alpha + v\cos\alpha$, where α is constant. Also let V be function of x and y. Then prove that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2}$.

VECTOR FUNCTIONS

When a particle moves through space during a time interval I, we think of the particle's coordinates as functions defined on I :

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in I. \quad \ldots (1)$$

The points $(x, y, z) = (f(t), g(t), h(t))$, $t \in I$, make up the **curve** in space that we call the particle's **path.** The equations and interval in Equation (1) **parametrize** the curve. A curve in space can also be represented in vector form. The vector

$$r(t) = \overrightarrow{OP} = f(t)i + g(t)j + h(t)k \quad \ldots (2)$$

from the origin to the particle's position $P(f(t), g(t), h(t))$ at time t is the particle's **position vector** (Fig. 1). The functions f, g and h are the **component functions (components)** of the position vector. We think of the particle's path as the **curve traced by** r during the time interval I. Fig. 1 displays several space curve generated by a computer graphing program. It would not be easy to plot these curves by hand.

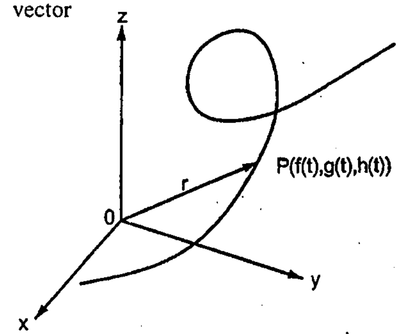

Fig. 1 : The position vector $r = \overrightarrow{OP}$ of a particle moving through space is a function of time

Equation (2) defines r as a vector function of the real variable t on the interval I. More generally, a **vector function** or **vector-valued function** on a domain set D is a rule that assigns a vector in space to each element in D. For now, the domains will be intervals of real numbers resulting in a space curve. If the domains will be regions in the plane. Vector functions will then represent surface in space. Vector functions on a domain in the plane or space also give rise to "vector fields", which are important to the study of the flow of a fluid, gravitional fields, and electromagnetic phenomena.

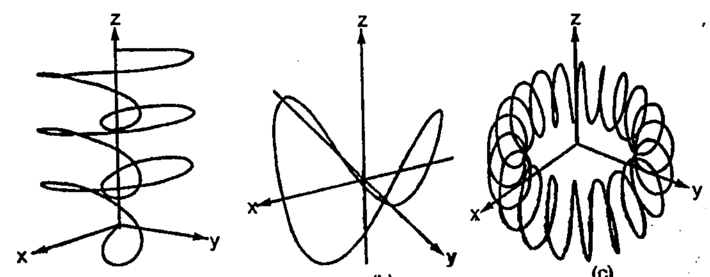

Fig. 2 : Computer-generated space curves are defined by the position vector $r(t)$

We refer to real-valued functions as **scalar functions** to distinguish them from vector nctions. The components of r are scalar functions of t. When we define a vector-valued nction by giving its component functions, we assume the vector function's domain to be : common domain of the components.

Example-1 : Graphing a Helix

Graph the vector function

$$r(t) = (\cos t)\, i + (\sin t)\, j + tk$$

$\mathbf{l^n}$. :

The vector function

$$r(t) = (\cos t)\, i + (\sin t)\, j + tk$$

defined for all real values of t. The curve traced by r a helix (from an old Greek word for "spiral") that winds und the circular cylinder $x^2 + y^2 = 1$ (Fig. 3). The curve ; on the cylinder because the i and j components of r, ng the x and y-coordinates of the tip of r, satisfy the inder's equation :

$$x^2 + y^2 = (\cos t)^2 + (\sin t)^2 = 1$$

The curve rises as the k component $z = t$ increases. h time t increases by 2π, the curve completes one turn und the cylinder. The equations

$$x = \cos t, \quad y = \sin t, \quad z = t$$

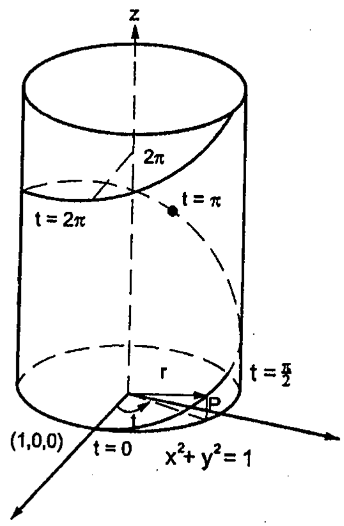

Fig. 3 : The upper half of the helix $r(t) = (\cos t)i + (\sin t)j + tk$

ımetrize the helix, the interval $-\infty < t < \infty$ being understood. You will find more helices 'ig. 4.

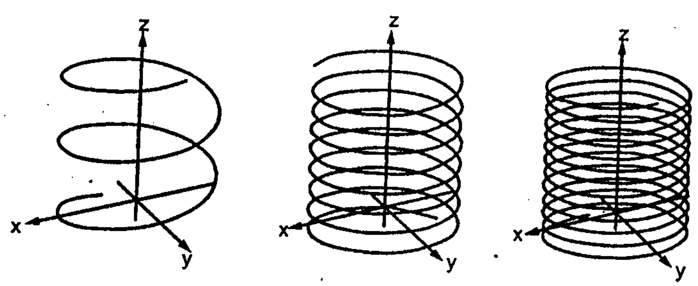

Fig. 4 : Helices drawn by computer

Limits and Continuity :

The way we define limits of vector-valued functions is similar to the way we define limits or real-valued functions.

Definition : Limit of Vector Functions :

Let $r(t) = f(t)i + g(t)j + h(t)k$ be a vector function and **L** a vector. We say that r has **limit L** as t approaches t_0 and write

$$\lim_{t \to t_0} r(t) = \mathbf{L}$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all t

$$0 < |t - t_0| < \delta \quad \Rightarrow \quad |r(t) - \mathbf{L}| < \epsilon.$$

If $\mathbf{L} = L_1 i + L_2 j + L_3 k$, then $\lim_{t \to t_0} r(t) = \mathbf{L}$ precisely when

$$\lim_{t \to t_0} f(t) = L_1, \quad \lim_{t \to t_0} g(t) = L_2, \quad \text{and} \quad \lim_{t \to t_0} h(t) = L_3$$

The equation

$$\lim_{t \to t_0} r(t) = \left(\lim_{t \to t_0} f(t)\right)i + \left(\lim_{t \to t_0} g(t)\right)j + \left(\lim_{t \to t_0} h(t)\right)k \quad \text{... (5)}$$

provides a practical way to calculate limits of vector functions.

■ **Example-2 :**

If $r(t) = (\cos t)\, i + (\sin t)\, j + tk$, then find $\lim_{t \to \pi/4} \bar{r}(t)$

Sol^n. :

$$\lim_{t \to \pi/4} r(t) = \left(\lim_{t \to \pi/4} \cos t\right)i + \left(\lim_{t \to \pi/4} \sin t\right)j + \left(\lim_{t \to \pi/4} t\right)k$$

$$= \frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2} j + \frac{\pi}{4} k.$$

We define continuity for vector functions the same way we define continuity for scalar functions.

Definition : Continuous at a Point

A vector function $r(t)$ is **continuous at a point** $t = t_0$ in its domain if $\lim_{t \to t_0} r(t) = r(t_0)$. The function is **continuous** if it is continuous at every point in its domain.

From Equation (3), we see that $r(t)$ is continuous at $t = t_0$ if and only if each component function is continuous there.

■ **Example-3 : Continuity of Space Curves**

(a) All the space curves shown in Fig. 2 and 4 are continuous because their component functions are continuous at every value of t in $(-\infty, \infty)$.

(b) The function

$$g(t) = (\cos t)\, i + (\sin t)\, j + \lfloor t \rfloor k$$

is discontinuous at every integer, where the greatest integer function $\lfloor t \rfloor$ is discontinuous.

Derivatives and Motion :

Suppose that $r(t) = f(t)\, i + g(t)\, j + h(t)\, k$ is the position vector of a particle moving along a curve in space and that f, g and h are differentiable functions of t. Then the difference between the particle's positions at time t and time $t + \Delta t$ is

$$\Delta r = r(t + \Delta t) - r(t)$$

Fig. 5(a). In terms of components,

$$\begin{aligned} \Delta r &= r(t + \Delta t) - r(t) \\ &= [f(t + \Delta t)\, i + g(t + \Delta t)\, j + h(t + \Delta t)\, k] - [f(t) i + g(t) j + h(t) k] \\ &= [f(t + \Delta t) - f(t)]\, i + [g(t + \Delta t) - g(t)]\, j + [h(t + \Delta t) - h(t)]\, k \end{aligned}$$

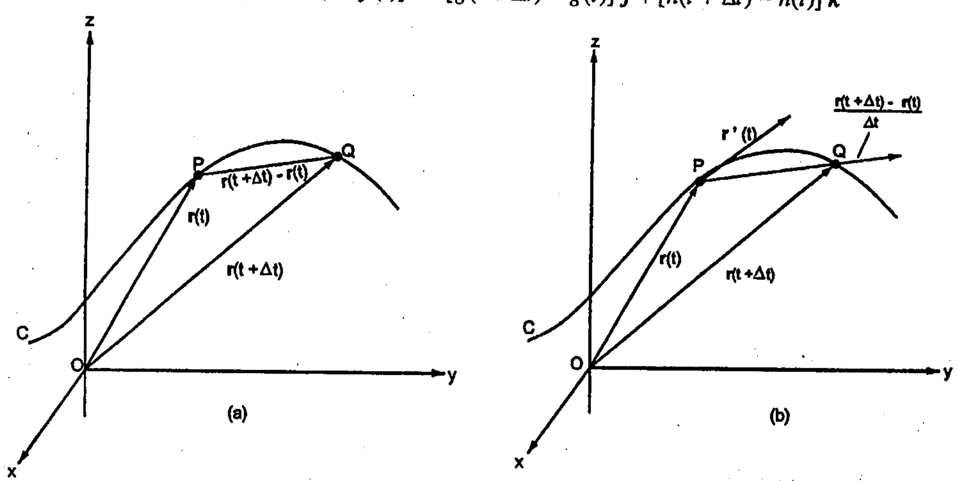

Fig. 5 : As $\Delta t \to 0$, the point Q approaches the point P along the curve C. In the limit, the vector $\overrightarrow{PQ}/\Delta t$ becomes the tangent vector $r'(t)$

As Δt approaches zero, three things seem to happen simultaneously. First, Q approaches P along the curve. Second, the secant line PQ seems to approach a limiting position tangent to the curve at P. Third, the quotient $\Delta r/\Delta t$ (Fig. 5(b)) approaches the limit

$$\lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \left[\lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}\right] i + \left[\lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}\right] j + \left[\lim_{\Delta t \to 0} \frac{h(t + \Delta t) - h(t)}{\Delta t}\right] k$$

$$= \left[\frac{df}{dt}\right] i + \left[\frac{dg}{dt}\right] j + \left[\frac{dh}{dt}\right] k$$

We are therefore led by past experience to the following definition.

Definition : Derivative

The vector function $r(t) = f(t)\, i + g(t) j + h(t) k$ has a **derivative (is differentiable) at t** if f, g and h have derivatives at t. The derivative is the vector function

$$r'(t) = \frac{dr}{dt} = \lim_{\Delta t \to 0} \frac{r(t + \Delta t) - r(t)}{\Delta t} = \frac{df}{dt} i + \frac{dg}{dt} j + \frac{dh}{dt} k.$$

A vector function r is **differentiable** if it is differentiable at every point of its domain. The curve traced by r is **smooth** if dr/dt is continuous and never **0**, that is, if f, g and h have continuous first derivatives that are not simultaneously 0.

The geometric significance of the definition of derivative is shown in Fig. 5. The points P and Q have position vectors $r(t)$ and $r(t + \Delta t)$, and the vector $\overrightarrow{PQ}$ is represented by $r(t + \Delta t) - r(t)$. For $\Delta t > 0$, the scalar multiple $(1/\Delta t)\, (r(t + \Delta t) - r(t))$ points in the same direction as the vector $\overrightarrow{PQ}$. As $\Delta t \to 0$, this vector approaches a vector that is tangent to the curve at P (Fig. 5(b)). The vector $r'(t)$, when different from **0**, is defined to be the vector **tangent** to the curve at P. The **tangent line** to the curve at a point $(f(t_0), g(t_0), h(t_0))$ is defined to be the line through the point parallel to $r'(t_0)$. We require $dr/dt \neq$ **0** for a smooth curve to make sure the curve has a continuously turning tangent at each point. On a smooth curve, there are no sharp corners or cups.

A curve that is made up of a finite number of smooth curves pieced together in a continuous fashion is called **piecewise smooth** (Fig. 6).

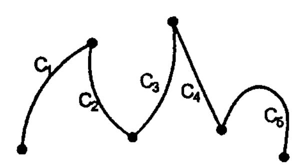

Fig. 6 : A piecewise smooth curve made up of five smooth curves connected end to end in continuous fashion

Look once again at Fig. 5. We drew the figure for Δt positive, so Δr points forward, in he direction of the motion. The vector $\Delta r/\Delta t$, having the same direction as Δr, points forward oo. Had Δt been negative, Δr would have pointed backward, against the direction of motion. The quotient $\Delta r/\Delta t$, however, being a negative scalar multiple of Δr, would once again have pointed forward. No matter how Δr points, $\Delta r/\Delta t$ points forward and we expect the vector

$dr/dt = \lim_{\Delta t \to 0} \Delta r/\Delta t$, when different from **0**, to do the same. This means that the derivative $\Delta r/\Delta t$ is just what want for modeling a particle's velocity. It points in the direction of motion and gives the rate of change of position with respect to time. For a smooth curve, the velocity s never zero; the particle does not stop or reverse direction.

Definition : Velocity, Direction, Speed, Acceleration

If r is the position vector of a particle moving along a smooth curve in space, then

$$v(t) = \frac{dr}{dt}$$

s the particle's **velocity vector**, tangent to the curve. At any time t, the direction of v is the direction of motion, the magnitude of v is the particle's **speed**, and the derivative $a = dv/dt$, when it exists, is the particle's **acceleration vector**. In summary,

1. Velocity is the derivative of position : $v = \dfrac{dr}{dt}$
2. Speed is the magnitude of velocity : Speed = $|v|$
3. Acceleration is the derivative of velocity : $a = \dfrac{dv}{dt} = \dfrac{d^2r}{dt^2}$
4. The unit vector $v/|v|$ is the direction of motion at time t.

We can express the velocity of a moving particle as the product of its speed and direction :

$$\text{Velocity} = |v|\left(\frac{v}{|v|}\right) = \text{(speed) (direction).}$$

Now let's look at an example of an object moving along a (nonlinear) space curve.

Example-4 : Flight of a Hang Glider

A person on a hang glider is spiraling upward due to rapidly rising air on a path having position vector $r(t) = (3\cos t)i + (3\sin t)j + t^2k$. The path is similar to that of a helix and shown in Fig. 7 for $0 \le t \le 4\pi$. Find

(a) the velocity and acceleration vectors,

(b) the glider's speed at any time t,

(c) the times, if any, when the glider's acceleration is orthogonal to its velocity.

Sol^n. :

(a) $r = (3\cos t)i + (3\sin t)j + t^2k$

$$v = \frac{dr}{dt} = -(3\sin t)i + (3\cos t)j + 2tk$$

$$a = \frac{d^2r}{dt^2} = -(3\cos t)i - (3\sin t)j + 2k$$

(b) Speed is the magnitude of v :

$$|v(t)| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (2t)^2}$$
$$= \sqrt{9\sin^2 t + 9\cos^2 t + 4t^2}$$
$$= \sqrt{9 + 4t^2}$$

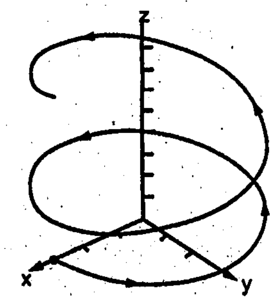

Fig. 7 : The path of a hang glider with position vector $r(t) = (3\cos t)i + (3\sin t)j + t^2k$

The glider is moving faster and faster as it rises its path.

(c) To find the times when v and a are orthogonal, we look for values of t for which

$$v \cdot a = 9\sin t\cos t - 9\cos t\sin t + 4t = 4t = 0$$

Thus, the only time the acceleration vector is orthogonal to v is when $t = 0$.

Differentiation Rules :

Because the derivatives of vector functions may be computed component by component, the rules for differentiating vector functions have the same form as the rules for differentiating scalar functions.

Differentiation Rules for Vector Functions :

Let u and v be differentiable vector functions of t, **C** a constant vector, c any scalar and f any differentiable scalar function.

1. Constant Function Rule : $\dfrac{d}{dt}\mathbf{C} = 0$
2. Scalar Multiple Rules : $\dfrac{d}{dt}[cu(t)] = cu'(t)$

$\dfrac{d}{dt}[f(t)u(t)] = f'(t)u(t) + f(t)u'(t)$

3. Sum Rule : $\dfrac{d}{dt}[u(t) + v(t)] = u'(t) + v'(t)$
4. Difference Rule : $\dfrac{d}{dt}[u(t) - v(t)] = u'(t) - v'(t)$

5. Dot Product Rule : $\frac{d}{dt}[u(t)\cdot v(t)] = u'(t)\cdot v(t) + u(t)\cdot v'(t)$

6. Cross Product Rule : $\frac{d}{dt}[u(t)\times v(t)] = u'(t)\times v(t) + u(t)\times v'(t)$

7. Chain Rule : $\frac{d}{dt}[u(f(t))] = f'(t)\, u'(f(t))$

When you use the Cross Product Rule, remember to preserve the order of the factors. If u comes first on the left side of the equation, it must also come first on the right or the signs will be wrong.

We will prove the product rules and Chain Rule but leave the rules for constants, scalar multiples, sums, and differences as exercises.

Proof of the Dot Product Rule :

Suppose that

$$u = u_1(t)i + u_2(t)j + u_3(t)k$$

and $$v = v_1(t)i + v_2(t)j + v_3(t)k$$

Then

$$\frac{d}{dt}(u\cdot v) = \frac{d}{dt}(u_1v_1 + u_2v_2 + u_3v_3)$$

$$= \underbrace{u_1'v_1 + u_2'v_2 + u_3'v_3}_{u'\cdot v} + \underbrace{u_1v_1' + u_2v_2' + u_3v_3'}_{u\cdot v'}$$

Proof of the Cross Product Rule :

We model the proof after the proof of the Product Rule for scalar functions. According to the definition of derivative,

$$\frac{d}{dt}(u\times v) = \lim_{h\to 0}\frac{u(t+h)\times v(t+h) - u(t)\times u(t)}{h}$$

To change this fraction into an equivalent one that contains the difference quotients for the derivatives of u and v, we subtract and add $u(t) \times v(t+h)$ in the numerator. Then

$$\frac{d}{dt}(u\times v) = \lim_{h\to 0}\frac{u(t+h)\times v(t+h) - u(t)\times v(t+h) + u(t)\times v(t+h) - u(t)\times v(t)}{h}$$

$$= \lim_{h\to 0}\left[\frac{u(t+h)-u(t)}{h}\times v(t+h) + u(t)\times\frac{v(t+h)-v(t)}{h}\right]$$

$$= \lim_{h\to 0}\frac{u(t+h)-u(t)}{h}\times\lim_{h\to 0}v(t+h) + \lim_{h\to 0}u(t)\times\lim_{h\to 0}\frac{v(t+h)-v(t)}{h}$$

The last of these equalities holds because the limit of the cross product of two vector functions is the cross product of their limits if the latter exist. As h approaches zero, $v(t+h)$ approaches $v(t)$ because v, being differentiable at t, is continuous at t. The two fractions approaches the values of du/dt and dv/dt at t. In short,

$$\frac{d}{dt}(u\times v) = \frac{du}{dt}\times v + u\times\frac{dv}{dt}.$$

Proof of the Chain Rule :

Suppose that $u(s) = a(s)\,i + b(s)\,j + c(s)\,k$ is a differentiable vector function of s and that $s = f(t)$ is a differentiable scalar function of t. Then a, b and c are differentiable functions of t, and the Chain Rule for differentiable real-valued functions gives

$$\frac{d}{dt}[u(s)] = \frac{da}{dt}i + \frac{db}{dt}j + \frac{dc}{dt}k$$

$$= \frac{da}{ds}\frac{ds}{dt}i + \frac{db}{ds}\frac{ds}{dt}j + \frac{dc}{ds}\frac{ds}{dt}k$$

$$= \frac{ds}{dt}\left(\frac{da}{ds}i + \frac{db}{ds}j + \frac{dc}{dt}k\right)$$

$$= \frac{ds}{dt}\frac{du}{ds}$$

$$= f'(t)\,u'(f(t)). \qquad (s = f(t))$$

Note : As an algebraic convenience, we sometimes write the product of a scalar c and a vector v as vc instead of cv. This permits us, for instance, to write the Chain Rule in a familiar form :

$$\frac{du}{dt} = \frac{du}{ds}\frac{ds}{dt}$$

where $s = f(t)$

Vector Functions of Constant Length :

When we track a particle moving on a sphere centered at the origin (Fig. 8), the position vector has a constant length equal to the radius of the sphere. The velocity vector dr/dt, tangent to the path of motion, is tangent to the sphere and hence perpendicular to r. This is always the case for a differentiable vector function of constant length : The vector and its first derivative are orthogonal. With the length constant, the change in the function is a change in direction only, and direction changes take place at right angles.

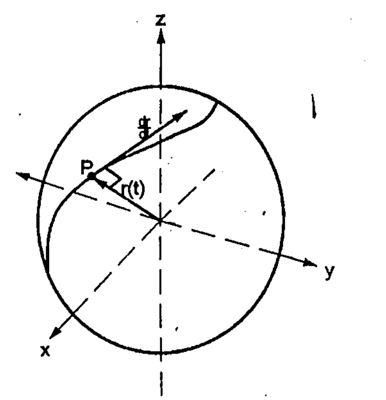

Fig. 8 : If a particle moves on a sphere in such a way that its position r is a differentiable function of time, then $r\cdot(dr/dt) = 0$

We can also obtain this result by direct calculation :

$$r(t)\cdot r(t) = c^2 \qquad |r(t)| = c \text{ is constant}$$

$$\frac{d}{dt}[r(t)\cdot r(t)] = 0 \qquad \text{Differentiate both sides}$$

$$r'(t)\cdot r(t) + r(t)\cdot r'(t) = 0 \qquad \text{Rule 5 with } r(t) = u(t) = v(t)$$

$$2r'(t)\cdot r(t) = 0$$

The vectors $r'(t)$ and $r(t)$ are orthogonal because their dot product is 0. In summary,

Ex. : If r is a differentiable vector function of t of constant length, then

$$r\cdot\frac{dr}{dt} = 0 \qquad \text{... (4)}$$

■ **Example-5** :

Show that $r(t) = (\cos t)\,i + \sqrt{5}\,j + (\sin t)\,k$ has constant length and is orthogonal to its derivative.

Sol^n. :

$$r(t) = (\cos t)\,i + \sqrt{5}\,j + (\sin t)\,k$$

$$|r(t)| = \sqrt{\cos^2 t + 5 + \sin^2 t} = \sqrt{6}$$

$$\frac{dr}{dt} = -\sin t\, i + \cos t\, k$$

$$r\cdot\frac{dr}{dt} = -\sin t\cos t + \sin t\cos t = 0$$

Integrals of Vector Functions :

A differentiable vector function $\mathbf{R}(t)$ is an **antiderivative** of a vector function $r(t)$ on interval I if $d\mathbf{R}/dt = r$ at each point of I. If $\mathbf{R}$ is antiderivative of r on I, it can be shown, working one component at a time, that every antiderivative of r on I has the form $\mathbf{R} + \mathbf{C}$ for some constant vector $\mathbf{C}$. The set of all antiderivatives of r on I is the indefinite integral of r on I.

Definition : Indefinite Integral

The **indefinite integral** of r with respect to t is the set of all antiderivatives of r, denoted by $\int r(t)\,dt$. If $\mathbf{R}$ is any antiderivative of r, then

$$\int r(t)\,dt = \mathbf{R}(t) + \mathbf{C}$$

The usual arithmetic rules for indefinite integrals apply.

■ **Example-6** : Finding Indefinite Integrals

$$\int((\cos t)i + j - 2tk)\,dt = \left(\int \cos t\,dt\right)i + \left(\int dt\right)j - \left(\int 2t\,dt\right)k \qquad \text{... (5)}$$

$$= (\sin t + C_1)\,i + (t + C_2)\,j - (t^2 + C_3)\,k \qquad \text{... (6)}$$

$$= (\sin t)i + t\,j - t^2k + \mathbf{C} \qquad \mathbf{C} = C_1 i + C_2 j - C_3 k$$

As in the integration of scalar functions, we recommend that you skip the steps in Equations (5) and (6) and go directly to the final form. Find an antiderivative for each component and add a constant vector at the end.

Definite integrals of vector functions are best defined in terms of components.

Definition : Definite Integral

If the components of $r(t) = f(t)i + g(t)j + h(t)k$ are integrable over $[a, b]$, then so is r, and the definite integral of r from a to b is

$$\int_a^b r(t)\,dt = \left(\int_a^b f(t)\,dt\right)i + \left(\int_a^b g(t)\,dt\right)j + \left(\int_a^b h(t)\,dt\right)k$$

■ **Example-7** : Evaluating Definite Integrals

$$\int_0^\pi((\cos t)\,i + j - 2tk)\,dt = \left(\int_0^\pi \cos t\,dt\right)i + \left(\int_0^\pi dt\right)j - \left(\int_0^\pi 2t\,dt\right)k$$

$$= [\sin t]_0^\pi\, i + [t]_0^\pi\, j - [t^2]_0^\pi\, k$$

$$= [0 - 0]\,i + [\pi - 0]\,j - [\pi^2 - 0^2]\,k$$

$$= \pi j - \pi^2 k$$

The Fundamental theorem of Calculus for continuous vector functions says that

$$\int_a^b r(t)\,dt = \mathbf{R}(t)]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where $\mathbf{R}$ is any antiderivative of r, so that $\mathbf{R}'(t) = r(t)$.

■ **Example-8** :

If acceleration vector of glider is $a(t) = -(3\cos t)\,i - (3\sin t)\,j + 2k$. We also know that initially (at time $t = 0$), the glider departed from the point (3, 0, 0) with velocity $v(0) = 3j$. Find the glider's position as a function of t.

Sol^n. :

Our goal is to find $r(t)$ knowing

The differential equation : $a = \dfrac{d^2r}{dt^2} = -(3\cos t)\,i - (3\sin t)\,j + 2k$

The initial conditions : $v(0) = 3j$ and $r(0) = 3i + 0j + 0k$

Integrating both sides of the differential equation with respect to t gives

$$v(t) = -(3\sin t)\,i + (3\cos t)\,j + 2tk + \mathbf{C}_1$$

We use $v(0) = 3j$ to find $\mathbf{C}_1$:

$$3j = -(3\sin 0)\,i + (3\cos 0)\,j + (0)k + \mathbf{C}_1$$

$$3j = 3j + \mathbf{C}_1$$

$$\mathbf{C}_1 = \mathbf{0}$$

The glider's velocity as a function of time is

$$\frac{dr}{dt} = v(t) = -(3\sin t)\,i + (3\cos t)\,j + 2t\,k$$

Integrating both sides of this last differential equation gives

$$r(t) = (3\cos t)\,i + (3\sin t)\,j + t^2 k + C_2$$

We then use the initial condition $r(0) = 3i$ to find C_2 :

$$3i = (3\cos 0)\,i + (3\sin 0)\,j + (0^2)k + C_2$$

$$3i = 3i + (0)\,j + (0)\,k + C_2$$

$$C_2 = 0.$$

The glider's position as a function of t is

$$r(t) = (3\cos t)\,i + (3\sin t)\,j + t^2 k$$

This is the path of the glider we know from Example-4 and is shown in Fig. 0.

MULTIPLE CHOICE QUESTIONS

■ **Fill in the Blanks :**

1. **Velocity is $\bar{v}$ = ______.**

(a) $\frac{dr}{dt}$ (b) $\frac{d\bar{r}}{dt}$ (c) $|v|$ (d) None

2. **Acceleration is $\bar{a}$ = ______.**

(a) $\frac{d^2r}{dt^2}$ (b) $\frac{d\bar{r}}{dt}$ (c) $\frac{d\bar{v}}{dt}$ (d) None

3. **Speed = ______.**

(a) $|\bar{v}|$ (b) $\bar{v}$ (c) $\bar{a}$ (d) None

4. **Speed of $\bar{r} = 3\cos t\,\bar{i} + 3\sin t\,\bar{j} + t^2\bar{k}$ is ______.**

(a) $\sqrt{9-4t^2}$ (b) $9-4t^2$ (c) $9+4t^2$ (d) $\sqrt{9+4t^2}$

5. **If $\bar{r}$ is a differentiable vector function of t of constant length, then ______.**

(a) $\bar{r}\cdot\frac{d\bar{r}}{dt}=0$ (b) $\bar{r}\cdot\frac{d\bar{r}}{dt}\neq 0$ (c) $\bar{r}\,\frac{d\bar{r}}{dt}=0$ (d) None

ANSWERS

1. (b), 2. (c), 3. ( ), 4. (d), 5. (a)

EXERCISE

Motion in the *xy*-plane :

In Exercise 1–4, $r(t)$ is the position of a particle in the *xy*-plane at time t. Find an equation in x and y whose graph is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of t.

1. $r(t) = (t+1)\,i + (t^2-1)\,j,\ t = 1$
2. $r(t) = (t^2+1)\,i + (2t-1)\,j,\ t = 1/2$
3. $r(t) = e^t i + \frac{2}{9}e^{2t} j,\ t = \ln 3$
4. $r(t) = (\cos 2t)\,i + (3\sin 2t)\,j,\ t = 0$

Exercise 5–8 give the position vectors of particles moving along various curves in the xy-plane. In each case, find the particle's velocity and acceleration vectors at the stated times and sketch them as vectors on the curve.

5. **Motion on the circle $x^2 + y^2 = 1$**

$r(t) = (\sin t)\,i + (\cos t)\,j;\ t = \frac{\pi}{4}$ and $\frac{\pi}{2}$

6. **Motion on the circle $x^2 + y^2 = 16$**

$r(t) = \left(4\cos\frac{t}{2}\right)i + \left(4\sin\frac{t}{2}\right)j;\ t = \pi$ and $\frac{3\pi}{2}$

7. **Motion on the cycloid $x = t - \sin t$, $y = 1 - \cos t$**

$r(t) = (t-\sin t)\,i + (1-\cos t)\,j;\ t = \pi$ and $\frac{3\pi}{2}$

8. **Motion on the parabola $y = x^2 + 1$**

$r(t) = t\,i + (t^2+1)\,j;\ t = -1, 0$ and 1

Velocity and Acceleration in Space :

In Exercises 9–14, $r(t)$ is the position of a particle in space at time t. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of t. Write the particle's velocity at that time as the product of its speed and direction.

9. $r(t) = (t+1)\,i + (t^2-1)\,j + 2t\,k,\ t = 1$
10. $r(t) = (1+t)\,i + \frac{t^2}{\sqrt{2}}\,j + \frac{t^3}{3}\,k,\ t = 1$
11. $r(t) = (2\cos t)\,i + (3\sin t)\,j + 4t\,k,\ t = \frac{\pi}{2}$
12. $r(t) = (\sec t)\,i + (\tan t)\,j + \frac{4}{3}t\,k,\ t = \frac{\pi}{6}$
13. $r(t) = (2\ln(t+1))\,i + t^2 j + \frac{t^2}{2}\,k,\ t = 1$
14. $r(t) = (e^{-t})\,i + (2\cos 3t)\,j + (2\sin 3t)\,k,\ t = 0$

In Exercises 15–18, $r(t)$ is the position of a particle in space at time t. Find the angle between the velocity and acceleration vectors at time $t = 0$. $\left[\cos\theta = \frac{\bar{v}\cdot\bar{a}}{|\bar{v}|\,|\bar{a}|}\right]$

15. $r(t) = (3t+1)\,i + \sqrt{3}t\,j + t^2 k$
16. $r(t) = \left(\frac{\sqrt{2}}{2}t\right)i + \left(\frac{\sqrt{2}}{2}t - 16t^2\right)j$
17. $r(t) = (\ln(t^2+1))\,i + (\tan^{-1}t)\,j + \sqrt{t^2+1}\,k$
18. $r(t) = \frac{4}{9}(1+t)^{3/2}\,i + \frac{4}{9}(1-t)^{3/2}\,j + \frac{1}{3}t\,k$

In Exercises 19 and 20, $r(t)$ is the position vector of a particle in space at time t. Find the time or times in the given time interval when the velocity and acceleration vectors are orthogonal.

19. $r(t) = (t-\sin t)\,i + (1-\cos t)\,j;\ 0 \le t \le 2\pi$
20. $r(t) = (\sin t)\,i + t\,j + (\cos t)\,k,\ t \ge 0$

Integrating Vector-Valued Functions :

Evaluate the integrals in Exercises 21–26.

21. $\int_0^1 (3t^2 i + 2j + (t-3)k)\,dt$
22. $\int_1^2 \left[(6-6t)\,i + 3\sqrt{t}\,j + \left(\frac{4}{t^2}\right)k\right]dt$
23. $\int_{-\pi/4}^{\pi/4} [(\sin t)\,i + (1+\cos t)\,j + (\sec^2 t)\,k]\,dt$
24. $\int_0^{\pi/3} [(\sec t\tan t)\,i + (\tan t)\,j + (2\sin t\cos t)\,k]\,dt$

5. $\int_1^4 \left[\frac{1}{t} i + \frac{1}{5-t} j + \frac{1}{2t} k\right] dt$

6. $\int_0^1 \left[\frac{2}{\sqrt{1-t^2}} i + \frac{\sqrt{3}}{1+t^2} k\right] dt$

nitial Value Problems for Vector-Valued unctions :

olve the initial value problems in Exercises 7–32 for r as a vector function of t.

7. Differential equation : $\frac{dr}{dt} = -t i - t j - t k$

Initial condition : $r(0) = i + 2j + 3k$

8. Differential equation :

$$\frac{dr}{dt} = (180t) i + (180t - 16t^2) j$$

Initial condition : $r(0) = 100 j$

9. Differential equation :

$$\frac{dr}{dt} = \frac{3}{2}(t+1)^{1/2} i + e^{-t} j + \frac{1}{t+1} k$$

Initial condition : $r(0) = k$

). Differential equation :

$$\frac{dr}{dt} = (t^3 + 4t) i + t j + 2t^2 k$$

Initial condition : $r(0) = i + j$

. Differential equation : $\frac{d^2 r}{dt^2} = -32k$

Initial conditions :

$$r(0) = 100\,k \text{ and } \left.\frac{dr}{dt}\right|_{t=0} = 8i + 8j$$

. Differential equation : $\frac{d^2 r}{dt^2} = -(i + j + k)$

Initial conditions :

$$\bar{r}(0) = 10\bar{i} + 10\bar{j} + 10\bar{k} \text{ and } \left.\frac{dr}{dt}\right|_{t=0} = 0$$

Tangent Lines to Smooth Curves :

As mentioned in the text, the tangent line to a smooth curve $r(t) = f(t) i + g(t) j + h(t) k$ is the line that passes through the point $(f(t_0), g(t_0), h(t_0))$ parallel to $v(t_0)$, the curve's velocity vector at t_0. $\bar{r}'(t)$ at $t = t_0$ is equation of tangent line at $t = t_0$. In Exercises 33–36, find parametric equations for the line that is tangent to the given curve at the given parameter value $t = t_0$.

33. $r(t) = (\sin t) i + (t^2 - \cos t) j + e^t k, t_0 = 0$

34. $r(t) = (2\sin t) i + (2\cos t) j + 5tk, t_0 = 4\pi$

35. $r(t) = (a \sin t) i + (a \cos t) j + btk, t_0 = 2\pi$

36. $r(t) = (\cos t) i + (\sin t) j + (\sin 2t) k, t_0 = \frac{\pi}{2}$

37. Each of the following equations in parts (a) – (e) describes the motion of a particle having the same path, namely the unit circle $x^2 + y^2 = 1$. Although the path of each particle in parts (a) – (e) is the same, the behavior, or "dynamics", of each particle is different. For each particle, answer the following questions.

(1) Does the particle have constant speed ? If so, what is its constant speed ?

(2) Is the particle's acceleration vector always orthogonal to its velocity vector ?

(3) Does the particle move clockwise or counterclockwise around the circle ?

(a) $r(t) = (\cos t) i + (\sin t) j, t \geq 0$

(b) $r(t) = \cos(2t) i + \sin(2t) j, t \geq 0$

(c) $r(t) = \cos\left(t - \frac{\pi}{2}\right) i + \sin\left(t - \frac{\pi}{2}\right) j, t \geq 0$

(d) $r(t) = (\cos t) i - (\sin t) j, t \geq 0$

(e) $r(t) = \cos(t^2) i + \sin(t^2) j, t \geq 0$

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SARDAR PATEL UNIVERSITY

QUESTION PAPER

OCTOBER, 2018

Time : 3 Hours] **[Total Marks : 70**

1. **Answer the following by selecting correct choice from the options :** **[10]**

(1) **If $y = \log(ax + b)$ then y_n = ______.**

(a) $\frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$ (b) $\frac{(-1)^{n-1}(n-1)! a^n}{(ax+b)^n}$

(c) $\frac{(-1)^n n! a^n}{(ax+b)^n}$ (d) $\frac{(-1)^{n-1} n! a^n}{(ax+b)^n}$

(2) **$\cosh x + \sinh x$ = ______.**

(a) e^x (b) e^{-x} (c) 1 (d) -1

(3) **Directrix of $y^2 = 10x$ is ______.**

(a) $x = \frac{5}{2}$ (b) $x = -\frac{5}{2}$ (c) $x = 10$ (d) $x = -10$

(4) **Asymptotes of $y = x^3 - 3x^2 - 2x$ are ______.**

(a) $x = 0, 1, 2; y = 1$ (b) $x = 0, -1, 2; y = 1$

(c) $x = 0, -1, 2$ (d) Not possible

(5) **The curve of $r = \cos 3\theta$ is symmetric about ______.**

(a) Polar axis (b) Normal axis

(c) Pole (d) Polar axis, Normal axis and Pole

(6) **If eccentricity $e = 1$ then conic is ______.**

(a) Hyperbola (b) Ellipse (c) Circle (d) Parabola

(7) **$\int_0^{\frac{\pi}{2}} \sin^{10} x\, dx$ = ______.**

(a) $\frac{63}{256}$ (b) $\frac{63}{512}$ (c) $\frac{63\pi}{256}$ (d) None

(8) If $J_{p,q} = \int_0^{\frac{\pi}{2}} \sin^p x \cos^q x \, dx$ then $J_{p,q}$ = ______.

(a) $\frac{p+1}{p+q} J_{p-2,q}$ (b) $\frac{p-1}{p+q} J_{p+2,q}$

(c) $\frac{p}{p-q} J_{p,q}$ (d) $\frac{p-1}{p+q} J_{p-2,q}$

(9) If $r = a(1 - \cos\theta)$ then ρ^2 = ______.

(a) $\frac{ar}{9}$ (b) $\frac{8r}{9}$ (c) $\frac{8ar}{9}$ (d) r

(10) Velocity is $\overline{V}$ = ______.

(a) $\frac{dr}{dt}$ (b) $\frac{d\bar{r}}{dt}$ (c) $|v|$ (d) None

2. Answer any TEN of the following : [20]

(1) If $y = \cos 3x$ then find y_4.

(2) Evaluate $\lim_{x \to a} (a - x) \tan\left(\frac{5\pi x}{2a}\right)$.

(3) If $y = e^{2x} \sin 5x$ then find y_n.

(4) Find the parametric equation for $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

(5) Discuss symmetries of the curve $xy - 16 = 0$.

(6) Transfer the equation $r = \tan\theta + \sec\theta$ in cartesian form.

(7) Evaluate $\int_0^1 x^5 \sin^{-1} x \, dx$.

(8) The region bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid by shell method.

(9) Find the area of the surface swept out by revolving the circle $x^2 + y^2 = 1, y > 0$ about x-axis.

(10) For the curve $y = a\sin 2x$ the find $\frac{ds}{dx}$.

(11) For $u = x^3 - 3xy^3$ then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

(12) Definition : Acceleration of a particle.

3. (a) State and prove Leibniz's theorem. [05

(b) Find a, b, c so that $\lim_{x \to 0} \frac{ae^x - 2b\cos x + 3ce^{-x}}{x \sin x} = 2$. [05

OR

3. (a) If $x = \cos\left(\frac{1}{m} \log y\right)$, then find $y_n(0)$. [05

(b) Find center - to - focus distance, foci vertices, center and asymptotes for the hyperbola $\frac{x^2}{4} - \frac{y^2}{5} = 1$. [05

4. (a) Discuss intercepts, symmetries, asymptotes, sign of function and hence sketch the curve $y = \frac{x^2 - 1}{x^2 - 4}$. [05

(b) State when a polar curve is symmetric with respect to polar axis ? Prove it. [05

OR

4. (a) Discuss symmetries, extent, closeness for the curve $r = 3(1 + \cos\theta)$. [05

(b) In usual notation prove that $r = \frac{pe}{1 \pm e \cos\theta}$. [05

5. (a) Obtain Reduction Formula for $\int_0^{\frac{\pi}{2}} \sin^n x \, dx$, where $n \in N$. [05

(b) The circle $x^2 + y^2 = a^2$ is rotated about the X-axis to generate the sphere find its volume. [05

OR

5. (a) Find the length of arc of the parabola $y^2 = 4ax$, $(a > 0)$, measured from the vertex to one extremity of its latusrectum. [05

(b) Evaluate : (1) $\int_0^{\frac{\pi}{4}} \cos^3 2x \sin^4 4x \, dx$ (2) $\int \tan^6 x \, dx$ [05

6. (a) For a polar equation $r = f(\theta)$ of an curve, prove that $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$. [05]

(b) In usual notation prove that $\rho = \frac{(1 + y_1^2)^{\frac{3}{2}}}{y_2}$. [05]

OR

6. (a) State and Prove Euler's theorem for $z = f(x, y)$. [05]

(b) If $z = xy\, f\left(\frac{y}{x}\right)$ and z is constant, then prove that $\frac{f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)} = \frac{x\left[y + x\frac{dy}{dx}\right]}{y\left[y - x\frac{dy}{dx}\right]}$. [05]

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US01CMTH21
Correction on Page No.
1, 4, 6, 247, 252

US01CMTH22
Pg: 35, Ex. 3 $x \to 0$ $\frac{-----}{---- + x^4} = \frac{3}{40}$
Pg. 70 Ex. 2, $x = \frac{1}{t}$, $y = \frac{t^2 - 1}{t}$

US01CMTH22
Correction on pg.: 1, 5, 16, 70, 123, 124, 146,